Probability in Action

edited by

Bartosz Przysucha

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Contents

Bartosz Przysucha

Preface

Probability in Action vol. 4, like the previous parts of this series, is a collection of scientific papers on probabilistic and statistics issues used in mathematical, economic, financial sciences etc. These papers were written by employees of the Lublin University of Technology and their collaborators from other universities in Poland and Ukraine. In this volume, in addition to presenting the research problems of the University of Technology scientists, we would also like to make an attempt to describe a certain aspect of the scientific work of the creator of the *Probability in Action* series, Professor Tadeusz Banek. The reason for this is not only the plentifulness of Professor Banek's scientific work, but also some round jubilees that act like a kind of paper clip, and taking into account the quantity and quality of the Professor's work, rather like a binder. The first date is 1969 and a master's degree at the Gdańsk University of Technology, Faculty of Ship Technology and Institute of Fluid-Flow Machinery Polish Academy of Sciences, entitled "Geometric optimization of the nozzle-iris system in turbine speed control due to minimal static error". In 2019, 50 years will have passed since that date. The second one is a doctorate in 1975 at the AGH University of Science and Technology, Faculty of Electrical Engineering, Institute of Automation. The title of the thesis is "On the existence of almost-saddle points for continuous games in Banach spaces". In 2020, 45 years will have passed since that date. This date is also the beginning of Professor Banek's work at the Lublin University of Technology. Professor Tadeusz Banek received the title of Professor of Technical Sciences in 2004.

Professor Tadeusz Banek was the Head of the Quantitative Methods in Management Department at the Faculty of Management of the Lublin University of Technology in 2004–2016, and earlier in 1992–2002 he was the founder and the Head of the Operational Research Department.

He is the author of two books in Polish (*Risk Account, Estimation Disorders in Monitoring Systems*), and a co-author of several books in English on adaptive control, modelling and risk analysis and self-learning. He was repeatedly awarded by the Rector of the Lublin University of Technology for his scientific achievements. He is the author of almost 100 scientific publications. It would be impossible to recall and present them here. Attempts to summarize at least a part of this work

were undertaken by his two closest collaborators Prof. Wojciech Batko and Edward Kozłowski, PhD.

I personally owe Professor T. Banek inspiration in taking up scientific issues related to acoustics and cooperation with Prof. Wojciech Batko (AGH University of Science and Technology) which continues the cooperation – initiated by Prof. W. Batko and Prof. T. Banek – between the Faculty of Management of the Lublin University of Technology and the Department of Mechanics and Vibroacoustics of the AGH University of Science and Technology.

Wojciech Batko

The outcome of scientific cooperation with Professor Tadeusz Banek

1 Introduction

It is hard to disagree with the statement that scientific discussion is the driving force for achievements and development in almost every field. The confrontation of different views allows for a better understanding of the nature of identified problems, generating new research questions as well as quicker and better response to new research challenges, and their implementation. Such a situation was an impulse for our mutual contacts and long-term cooperation.

My first meetings and scientific discussions with Professor Tadeusz Banek began over twenty-five years ago and I hope that it will be continued bringing us satisfaction and measurable achievements in science. The cooperation was initiated in 1992 after the Professor defended the habilitation dissertation entitled it Optimal filtration and prediction of signals described by stochastic differential equations (Banek (1990)) at my home Faculty of Mechanical Engineering and Robotics at the AGH University of Science and Technology, and which dissertation I had the privilege of being a reviewer. We have had long discussions on the various conditions of the modelling process of dynamic behaviour of mechanical systems, including their description with stochastic differential equations, which allowed for a fuller understanding and limiting the field of mutual reservations.

Although none of us were completely satisfied, the meeting resulted in the determination of future themes worth taking, and subsequent long-term cooperation and mutual friendship. It has materialized itself in the implementation of two centrally-funded research projects (Banek and Batko (1991–1993, 1994–1996)).

In this paper I will outline a fragment of issues developed within our long-term cooperation documented by a series of joint publications (Banek and Batko (1992a, 1992b, 1993a, 1993b, 1993c, 1994a, 1994b, 1994c, 1995, 1996a, 1996b, 1997a, 1997b, 1997c, 1997d), including a monograph (Banek and Batko (1997e)).

2 Optimal filtration and prediction – new tools in the construction of systems monitoring changes in hydrodynamic bearing nodes as a result of cooperation with Prof. Tadeusz Banek

2.1 Rotary machine monitoring systems

Vibration monitoring systems for their bearing nodes are an integral part of multi-channel condition monitoring systems. One of the typical elements of such solutions, widely used in industry, is a module for controlling the relative vibrations of the shaft and movements of the shaft journals in a hydrodynamic plain bearing.

Schematic representation of their implementation in relation to systems monitoring the condition of the turbine set is illustrated in Fig. 1.

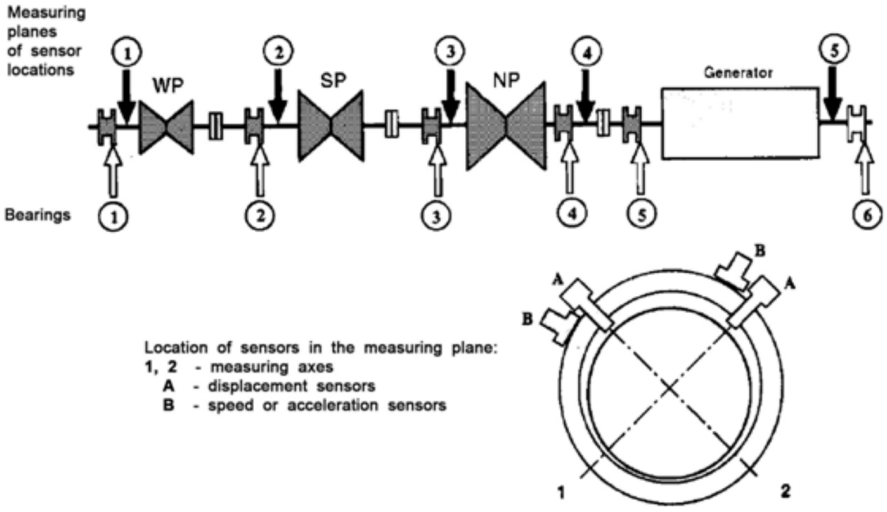

Figure 1. Schematic diagram of the turbine set vibration monitoring system

Their implementation is determined by continuous measurement of shaft journal displacement in the bearing, in two mutually perpendicular directions, carried out with eddy current sensors (as illustrated in Figure 2).

The process of monitoring the change in the bearing node condition (Figure 3) determine the reference of the measurement of the maximum radius of the journal vibrations on its trajectory, or alternatively, the value of the higher peak-to-peak amplitude of the vibration displacements from both controlled signals, collected during the period associated with the shaft rotational frequency, to the appropriate criterion values.

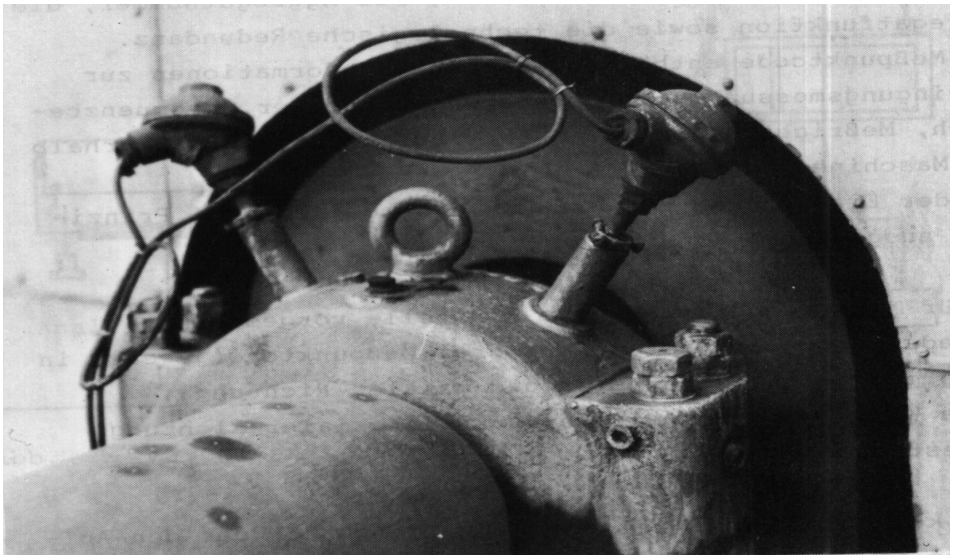

Figure 2. Installation of measuring sensors on the monitored machine plain bearings

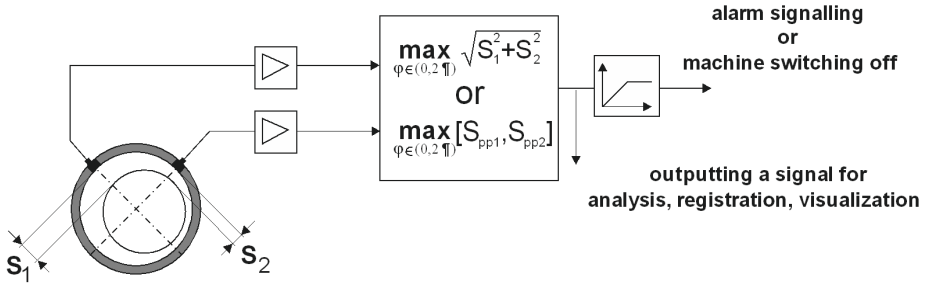

Figure 3. Vibration monitoring system for sliding bearing nodes

Exceeding the permissible change ranges by the monitored vibration waveforms, are associated with the possibility of stability loss of the equilibrium position and the emergence of self-excited vibrations with a large amplitude, which may cause breakage of the oil bearing layer and destruction of the bearing.

The eddy current sensors used in these measurement modules are very sensitive to ferromagnetic surface inhomogeneities of the monitored shaft, as well as various types of damage, for example, rifts, scratches, deformations or residues occurring on their surfaces. The mentioned impacts are referred to as electrical and mechanical runout. Their presence leads to significant measurement distortions. In order to minimize them, solutions are needed to filter out the interference they generate.

The proposed solution for filtration of monitored diagnostic signals (examined and tested at the IMP PAN in Gdańsk thanks to establishing cooperation with Prof. J. Kiciński), was a solution based on the Kalman filtration algorithm developed jointly with Professor Tadeusz Banek. It naturally incorporates into the idea of building a monitoring system associated with the dynamic model of a rotor machine supported by 2 hydrodynamic bearing nodes and measurement equations determined by the adopted measurement instrumentation present in the monitoring system.

2.2 The algorithm of optimal filtration and prediction in the system of monitoring the dynamics movement of the shaft journal in the hydrodynamic plain bearing

The task of eliminating disturbances in the shaft journal vibrations monitoring system in the hydrodynamic bearing can be considered through the prism of the measurement system performing the observation of dynamical system behaviour: journal – bearing shell – external restraints, being part of a dynamic system "rotor – bearings – supports – foundation". The probabilistic structure of such a description is defined by disturbances related to disturbances in the measurement path, as well as by the influence of factors omitted in the model description. Such an approach to the considered task allows to focus attention on searching for the optimal filtration and forecasting solution in Kalman's perspective. In the solutions of this group, it is assumed that the unobservable dynamic state of the considered diagnostic process $\underline{x}_t$ is described with accuracy to a certain Gaussian process. It is observable with measuring noise through specific control variables $\underline{y}_t$ mutually conjugated with them.

In this approach, the problem of filtration and prediction is equivalent to finding a solution in which the best medium square variable estimator $\underline{x}_t$ is the conditional expected value $m_t = E\left[\underline{x}_t|\underline{y}_t\right]$.

An explicit and effective solution to this problem created by Kalman-Bucy (see Anderson and Moor (1984)) – assuming that the distribution process $\left(\underline{x}_t|\underline{y}_t\right)$ is normal – is based on the assumption that the considered processes are generated by a stochastic system of differential equations:

$$\begin{aligned} d\underline{x}_t &= (A\underline{x}_t + U_t)\,dt + dv_{1t}, \qquad \underline{x}_0 = \underline{x} \\ d\underline{y}_t &= H\underline{x}_t dt + dv_{2t}, \qquad \underline{y}_0 = 0 \end{aligned} \tag{1}$$

in which $\underline{x}_t$ is an unobserved state vector of the monitored system, $\underline{y}_t$ – a vector of control observations, and U_t – a vector of forces acting on the system in the

presence of v_{1t}, v_{2t} – an independent pair of interfering Wiener processes. The implementation of such a task requires the reference of the adopted mathematical abstraction to the diagnostic model of the controlled process, providing a diagnostic interpretation for the A, H and U matrices (i.e. structural connections between particular parts of the adopted model) and interfering processes v_{1t}, v_{2t} .

When assessing the usefulness of this mathematical solutions group, it should be noted that they are characterized by a fully formalized approach. In a consistent way, from one model base, it allows recognizing possible dynamic states of diagnosed objects as well as the implementation of disturbance filtration tasks overlapping the observation of controlled diagnostic signals and the prediction of changes in their states.

In the realization of the proposed idea of interference suppression process in systems monitoring journal vibrations in a hydrodynamic bearing, the simplified linear model shown in Fig. 4 was used.

The model consists of an unbalanced rotor with elasticity c, massless, symmetrical with added mass m concentrated in the middle, supported on two identical plain bearings where sliding bearing parameters are: viscosity L, diameter of the shaft journal D, radial clearance ΔR and dynamic viscosity μ_0 of the existing oil.

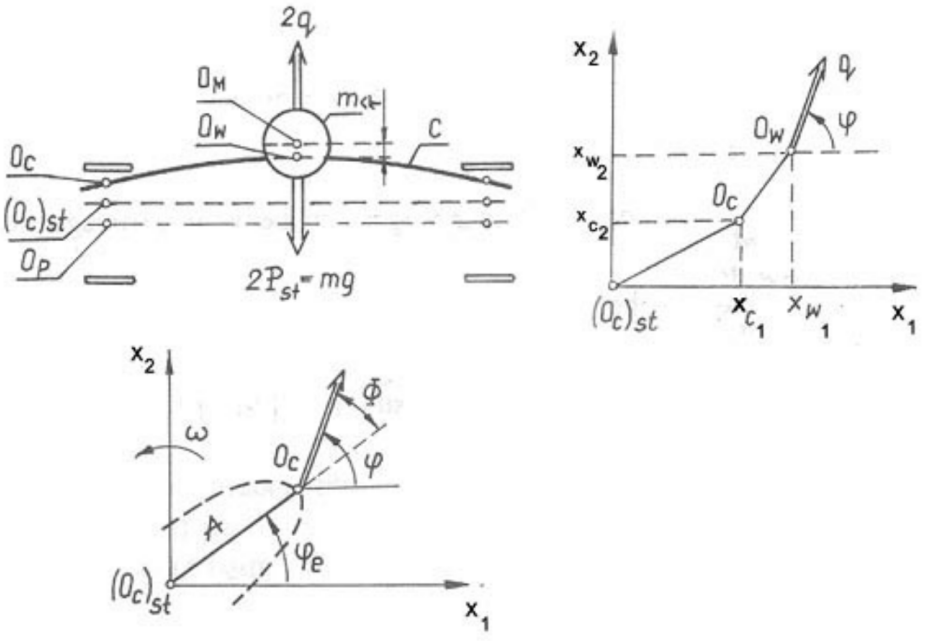

Figure 4. "Rotor-Bearing" system model

In its formulation, the equations of the rotor's centre of mass movement O_w take the form of:

$$\begin{aligned} -m\ddot{x}_{w1} - c(x_{w1} - x_1) + 2q_1 = 0 \\ -m\ddot{x}_{w2} - c(x_{w2} - x_2) + 2q_2 = 0 \end{aligned} \tag{2}$$

while the description of the shaft journal movement around a stable equilibrium position (in the hydrodynamic bearing) is given by the following formulas:

$$\begin{aligned} 2\Delta W_1 + c(x_{w1} - x_1) = 0 \\ 2\Delta W_2 + c(x_{w2} - x_2) = 0 \end{aligned} \tag{3}$$

where x_{w1}, x_{w2} indicate the rotor centre movement coordinates, x, y – the journal movement coordinates, q_1, q_2 – components of external load caused by unbalance, ΔW_1, ΔW_2 – components of the increase of the oil film dynamic reaction of slide bearings are given by the following formulas:

$$\begin{aligned} \Delta W_1 = W_1 - (W_1)_{st} = c_{11}x_1 + c_{12}x_2 + d_{11}\dot{x}_1 + d_{12}\dot{x}_2 \\ \Delta W_2 = W_2 - (W_2)_{st} = c_{21}x_1 + c_{22}x_2 + d_{21}\dot{x}_1 + d_{22}\dot{x}_2 \end{aligned} \tag{4}$$

determine by stiffness $c_{i,k}$ and damping $d_{i,k}$ coefficients of the oil film.

Starting from the above vibration equations of the shaft journal in the plain bearing (transformed into dimensionless variables specified by dimensionless time $\tau = \omega t$, associated with angular speed of the rotor ω, [rad/s] and displacements of the journal center related to the radial clearanceΔR) the dynamics of the monitored bearing node can be determined by the state vector

$$X = \begin{pmatrix} X_1 \\ \dot{X}_1 \\ \ddot{X}_1 \\ X_2 \\ \dot{X}_2 \\ \ddot{X}_2 \end{pmatrix}.$$

Assuming that modelling inaccuracies can be included in the disturbances description in the form of a Gaussian process Ξ overlapping with it, the following equation is obtained:

$$\dot{X} = AX + U + \Xi \tag{5}$$

The A and U matrices appearing in its description take the form of:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ c_{11}^* & \Omega & d_{11}^* & c_{12}^* & \Omega & d_{12}^* \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c_{21}^* & \Omega & d_{21}^* & c_{22}^* & \Omega & d_{22}^* \end{bmatrix}; \qquad U = \begin{bmatrix} 0 \\ 0 \\ Q_1 \\ 0 \\ 0 \\ Q_2 \end{bmatrix}. \tag{6}$$

The matrix coefficients c^*_{ij}, d^*_{ij} are related to the c_{ik} – stiffness and d_{ik} – damping coefficients of the oil film, through their reference to a dimensionless figure. Parameter

$$\Omega = -\frac{1}{\mu}\left(\frac{\omega_0}{\omega}\right)^2$$

is a reference parameter adopted in the normalization transformation to dimensionless time, specified as $\omega = \sqrt{g/\Delta R}$ and relative flexibility of the rotor $\mu = f/\Delta$, and Q_1, Q_2 are dimensionless (resulting from conducted standardization) components of the rotational force induced by the imbalance radius. Assuming disturbances, the accuracy of the adopted journal motion model (5) for state variables:

$$X = \begin{pmatrix} X_1 \\ \dot{X}_1 \\ \ddot{X}_1 \\ X_2 \\ \dot{X}_2 \\ \ddot{X}_2 \end{pmatrix}$$

in the form of a specific Gaussian process Ξ and its observations (7) with Gaussian measurement disturbances γ through selected components of the vector $Y = [X_1, 0, 0, X_2, 0, 0]$:

$$Y = HX + \gamma \tag{7}$$

we obtain the (5) and (7) equations. Their form is appropriate for the equations that generate the form of the optimal Kalman filter-predictor. The Ξ, γ and H matrices presented in the description are given by the following formulas:

$$\Xi = \sigma \begin{bmatrix} 0 \\ 0 \\ \text{white noise} \\ 0 \\ 0 \\ \text{white noise} \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \rho = \sigma \begin{bmatrix} \text{white noise} \\ \text{white noise} \end{bmatrix} \tag{8}$$

The searched Kalman filter equation for monitored diagnostic signals takes the form of:

$$d\hat{X}_t = A_t \hat{X}_t dt + \frac{R(t)H^T}{\rho^2}\left[d\hat{Y}_t - H_t \hat{X}_t dt\right] + U(t)dt \tag{9}$$

where $R(t)$ is the solution to the Riccati matrix equation

$$\dot{R}(t) = \sigma^2 I + AR + RA^T - RH^T \frac{1}{\rho^2} HR \tag{10}$$

determining the estimation error:

$$R(t) = E\left\{\left[\hat{X}_t - X_t\right]\left[\hat{X}_t - X_t\right]^T |\Gamma_t\right\} \tag{11}$$

defined on the Γ_t set,$\{y_1 = x_1, y_2 = x_2\}$ observation from the interval $(0,t)$. They determine the form of the algorithm for the module performing the process of filtration disturbances of controlled shaft journal vibrations in the bearing node by the monitoring system. Abandonment of the above formalization allows to solve the problem of prediction changes in controlled variables. The equations defining the predicator form $x(\tau,t)$ with fixed t and variable $\tau > 0$, for $\tau > 0$, $t > 0$ – estimated with a medium-square error – is given by the equation:

$$\frac{\partial X(\tau,t)}{\partial t} = AX(\tau,t) + U(\tau); \quad X(\tau,t) = \hat{X}_t \tag{12}$$

The initial value specifying its estimator $X(0,t) = \hat{X}_t$ at the moment t, it is given by the output from the Kalman-Bucy filter, defined by equation (9).

2.3 Summary

Our cooperation with the Jubilarian – concerning the above discussed the interference elimination method in the process of monitoring shaft journal vibrations in the bearing of the slide bearing, defined by Kalman filtration model – has shown that the direction of our joint search for effective filtration tools and prediction of monitored diagnostic signals based on the Kalman's filtration and prediction theory turned out to be appropriate and prospective. Hardware implementation on signal processors, newly constructed monitoring systems has been found. It has the advantage of optimality of monitored signals processing, with the possibility of uncertainty estimation of realized processes. This solution is characterized by strict connections of controlled diagnostic phenomena with physical mechanisms of monitored diagnostic signals generation.

The outlined and tested solutions are an up-to-date guide for designers of monitoring systems innovative constructions. It fully corresponds to the development of modern technologies for prototyping intelligent monitoring systems based on signal processors, which determines the place of their new implementations in the new generation of monitoring systems.

3 Conclusion

In the article I outlined only certain research area which connected us with the Jubilarian, inspired and motivated to work. It is a fragment of what we managed to achieve together, cooperating in the analysis of various vibroacoustic issues modelled with stochastic differential equations, in which area Professor Tadeusz Banek

is an authority, and of the results of which we can be proud in the environment of people dealing with technical diagnostics. It would not be possible without the Jubilarian's openness to cooperate in solving many of our problems, as well as encouraging his colleagues to cooperate with our Department of Mechanics and Vibroacoustics of the AGH University of Science and Technology.

I hope that we will be able to continue our cooperation. It is not a slogan, it is a dream. Professor is close to the employees of the Department of Mechanics and Vibroacoustics at the AGH University of Science and Technology and has a great deal of sympathy.

This results not only from the scientific authority of the Jubilarian, but also to the conviction that he is a friend of our department who follows the scientific development of the Kraków scientific community, shares his knowledge and is always helpful in our research work.

References. List of publications defining the results of cooperation with Professor Tadeusz Banek

Anderson B.D., Moor J.B., (1984), *Filtracja optymalna*, WNT, Warszawa.

Banek T., (1990), *Optymalna filtracja i predykcja sygnałów opisywanych stochastycznymi równaniami różniczkowymi*, Wydawnictwo Uniwersytetu Marii Curie-Skłodowskiej, Lublin.

Banek T., Batko W., (1992a), *Ein stochastisches Modell von Nachhallvorgangen zur Bestimmung der Nachhallzeit*, Proceedings of FASE-CONGRESS 1992, Zurich, 113–116.

Banek T., Batko W., (1992b), *Zastosowanie filtru Kalmana w procesie monitorowania stanu węzłów łożyskowych*, Materiały Konferencji EXPLO-SIL'92.

Banek T., Batko W., (1993a), *Filtracja zakłóceń w procesie dozorowania łożysk hydrodynamicznych*, Zagadnienia Eksploatacji Maszyn, z. 4 (96).

Banek T., Batko W., (1993b), *Noise filtering in monitoring systems*, Journal of Applied Mathematics and Computer Science, vol. 3, no. 3, 509–517.

Banek T., Batko W., (1993c), *Optymalna filtracja i predykcja w procesach monitorowania stanu łożysk hydrodynamicznych*, Materiały VII Krajowego Sympozjum Eksploatacji Urządzeń Technicznych. Problemy Eksploatacji 6/93, t. Diagnostyka, 33–40.

Banek T., Batko W., (1994a), *Estymacja stanów alarmowych w systemach drganiowego nadzoru stanu łożysk ślizgowych*, Materiały XIV Ogólnopolskiej Konferencji Naukowo-Dydaktycznej Teorii Maszyn i Mechanizmów, Gdańsk.

Banek T., Batko W., (1994b), *Nowe rozwiązania filtracyjne i predykcyjne w systemach drganiowego monitoringu stanu łożysk ślizgowych*, ZN AGH. Mechanika, t. 13, z. 1, 33–45.

Banek T., Batko W., (1994c), *Method of supporting functionals in the estimation of alarm conditions in systems of vibrational monitoring*, Journal of Theoretical and Applied Mechanics, no. 4, vol. 32, 931–944.

Banek T., Batko W., (1995), *Analiza czasu wystąpienia zaburzeń w systemach monitorujących*, Materiały III Ogólnopolskiej Konferencji UKŁADY DYNAMICZNE W ASPEKCIE TEORII I ZASTOSOWAŃ, red. J. Awrajcewicz, J. Mrozowski, Łódź, 69–76.

Banek T., Batko W., (1996a), *Filtering of the noise generated by shaft neck contour deformations in monitoring systems*, Applied Mathematics and Computer Science, vol. 6, no. 1, 97–108.

Banek T., Batko W., (1996b), *Zastosowanie rozwinięcia Wienera-Ito do symulacyjnego wyznaczania czasu wyjścia monitorowanych sygnałów*, Zeszyty Naukowe Politechniki Rzeszowskiej, s. Mechanika, z. 48, 17–22.

Banek T., Batko W., (1997a), *Time Estimation of Disturbance Occurrences in Monitoring Systems*, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 77, 1–8.

Banek T., Batko W., (1997b), *Symulacyjna metoda analizy zjawisk zużyciowych na proces wyjścia monitorowanych przebiegów z obszarów dopuszczalnych*, Zeszyty Naukowe AGH, Mechanika, t. 15, z. 1, 13–22.

Banek T., Batko W., (1997c), *Optimal filtering of disturbances in the process of estimation of vibration spectrum in diagnostics signals boenig monitored,* Mechanical Systems and Signal Processing 11(7).

Banek T., Batko W., (1997d), *Symulacyjna analiza pracy systemu monitorującego*, Mat. XXIV Sympozjum Diagnostyka Maszyn. Pol. Śląska, z. 1, 27–36.

Banek T., Batko W., (1997e), *Estymacja zaburzeń w systemach monitorujących*, Wydawnictwo AGO, Kraków.

Banek T., Batko W., (1991–1993), *Opracowanie małowrażliwych na zakłócenia rozwiązań monitorujących*, Project of Ministry of Science and Higher Education no. 3 1157 91 01 PB 582/3/91.

Banek T., Batko W., (1994–1996), *Estymacja czasu wystąpienia stanów awaryjnych w systemach monitorujących*, Project of Committee of Scientific Research no. 3P 402 018 0.

Edward Kozłowski[1]

Optimal control with a learning problem of the stochastic linear system

Keywords: stochastic control, linear quadratic control, self-learning, conditional entropy

Abstract

This paper presents the problem of optimal control of the stochastic system with the possibility of enriching knowledge about the system's parameters, where the control objective is to minimize the total costs associated with costs of control and cost of learning. In this case a classical concept of adaptation was used, which is based on tuning of controller by using the self-learning idea. Additionally the similarities and differences of control for the stochastic systems with known and unknown parameter related to scaling of transformation matrix was presented.

1 Introduction

The goal of any scientific discussion is not only to exchange views, remarks and observations but also to put the right questions and research problems. The exchange of knowledge and experience is the base in the process of development of scientific community, member organizations and society. A propensity to learn the phenomena and dependencies which exist in the real world around us needs to put the right research issues and to find answers to them.

My cooperation with Professor Tadeusz Banek began in 1996 when I was employed at the Lublin University of Technology at the Faculty of Management and Fundamentals of Technology. At the beginning, the scope of scientific and research work concerned the estimation of information cost about return rates of financial instruments and possibility of purchase that information during the investment portfolio construction. The subject has been presented in Banek, Kowalik and Kozłowski (1999) and Banek and Kozłowski (2003a, 2003b). Since 2003 research works were focused on the problem of determining adaptive control for

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stochastic systems and topics related to learning and self-learning of systems by control (see e.g. Banek, Kozłowski (2006)). The main aim of the self-learning process is enrich our knowledge about the operation and behaviour of the system in different environments.

Definition 1 *Stochastic adaptive control is a method of controlling systems with unknown parameters or parameters that change over time, where these parameters are identified (adjusted, tuned) by the controller.*

The tasks of adaptive control theory focus on the analysis and mathematical modelling of systems and development of controllers to realise given objectives, see Zabczyk (1996). It turns out that not only system state is influenced on the optimal control. The knowledge of system parameters and the information about the length of time interval is necessary to optimally control the system (Banek, Kozłowski (2014), Kozłowski (2011, 2018)). In addition, both the lack of knowledge of the system parameters and the control horizon causes an additional costs incurred during the control (see e.g. Kozłowski (2010, 2013)).

This paper is a continuation of outcomes obtained during our long-term cooperation with Professor Tadeusz Banek and presented in a series of works (Banek, Kozłowski (2005, 2006, 2011, 2014)). The task presented in paper is an extension of classical problem of linear quadratic control, where the negative effect related to ignorance of system behaviour was additionally taken into account.

2 Problem formulation

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. On this space we define the sequence $\{w_i\}_{1\leq i\leq N}$ of independent random vectors $w_i : \Omega \to \mathbb{R}^m$ with a normal distribution $N(\bar{0}, I_m)$, where $\bar{0} \in \mathbb{R}^m$ is the zero vector, while $I_m \in \mathbb{R}^{m\times m}$ is an identity matrix. Let $\xi : \Omega \to \mathbb{R}$ will be a random vector with a priori distribution $N(m_0, s_0)$, while $y_0 : \Omega \to \mathbb{R}^n$ – the initial state with $P(dy_0)$ distribution. We assume that all the aforementioned objects are stochastically independent and define non-decreasing families of $\sigma-$fields $\{\mathscr{F}_i\}_{0\leq i\leq N}$ and $\left\{\mathscr{F}_i^{\xi}\right\}_{0\leq i\leq N}$, where $\mathscr{F}_i = \sigma\{y_0\} \vee \sigma\{w_s : s = 1,2,...,i\}$, $\mathscr{F}_i^{\xi} = \mathscr{F}_i \vee \sigma\{\xi\}$ and we assume $\mathscr{F} = \mathscr{F}_N^{\xi}$.

Below we consider the linear stochastic system is defined by the state equation

$$y_{i+1} = y_i + \xi B u_i + \sigma w_{i+1}, \tag{1}$$

where $i = 0,...,N-1$, $y_i \in \mathbb{R}^n$, $B : \mathbb{R}^n \times \mathbb{R}^l \longrightarrow \mathbb{R}^n$ and $\sigma \in \mathbb{R}^{n\times m}$.

The vector $u_i \in \mathbb{R}^l$, $0 \leq i \leq N-1$ measurable with respect to σ-field $\mathscr{F}_i$ is called a control action. This same object may be controlled in different environments. The random variable ξ represents a scale value of the control transition

matrix. For the fixed horizon N we control the system (1) at decision moments $i = 0,1,...,N-1$ and we undertake a sequence of control actions $\{u_i\}_{0\leq i\leq N-1}$, which are referred as admissible control. The class of admissible controls is denoted by U.

Let R and Q be a positive defined matrices. For any $u \in U$ the value $u^T Ru$ represents the cost due to control u, the value $(y_N - a)^T Q(y_N - a)$ means the losses that are associated with the lack of hitting the target $a \in \mathbb{R}^n$ at the moment N. In the classical task the objective function is defined as the sum of costs associated with the controls at the moments $0,1,...,N-1$ and costs due to lack of hitting the target. Thus the objective function can be defined as

$$J^N(u) = E\left(\sum_{i=0}^{N-1} u_i^T Ru_i + (y_N - a)^T Q(y_N - a)\right). \tag{2}$$

On the other hand process of learning about the parameters should be enabled during the control. In real world the learning process can generate additional costs. In many tasks the information about system parameters is measured by entropy or Fisher information value (Banek, Kozłowski (2005, 2006, 2011)). Below the entropy concept will be applied to measure our knowledge about unknown parameter in system (1).

Remark 2 *If the random variable $X : \Omega \to \mathbb{R}$ has a normal distribution $N(\mu, s^2)$, then the entropy is equal*

$$H(X) = ln\left(\sqrt{2\pi e}s\right). \tag{3}$$

From (3) we can notice, when we possess more information about random variable (the distribution is more concentrated and the realization may be precisely predicted) then the entropy is smaller. Thus, additionally by minimizing entropy of parameter ξ we can get more information about behavior of system (1).

The apriori distribution of the random variable ξ is normal $N(m_0, s_0)$. Applying the Kalman-Bucy filter for discrete time stochastic process $\{y_t\}_{0\leq t\leq N}$ (see e.g. Liptser, Shiryaev (1978), Saridis (1995)) we obtain:

1. the best estimator (mean square sense) of random variable ξ at the moment j is a conditional expected value with respect of σ-field $\mathscr{F}_j$;
2. the conditional distribution $P\left(d\xi \,\middle|\, \mathscr{F}_j\right)$ is a normal distribution $N(m_j, s_j)$, where the conditional expected value of random variable ξ

$$m_j = E\left(\xi \,\middle|\, \mathscr{F}_j\right)$$

and the conditional variance

$$s_j = E\left((\xi - m_j)^2 \,\middle|\, \mathscr{F}_j\right)$$

are expressed by formulas

$$m_j = \frac{m_0 + s_0 \sum_{i=0}^{j-1} u_i^T B^T \left(\sigma\sigma^T\right)^{-1} \left(y_{i+1} - y_i\right)}{1 + s_0 \sum_{i=0}^{j-1} u_i^T B^T \left(\sigma\sigma^T\right)^{-1} B u_i} \tag{4}$$

and

$$s_j = \frac{s_0}{1 + s_0 \sum_{i=0}^{j-1} u_i^T B^T \left(\sigma\sigma^T\right)^{-1} B u_i}. \tag{5}$$

Remark 3 *From (3) and (5) we have, that at the moment N (at the end of control interval) the conditional entropy of random variable ξ is equal*

$$H\left(\xi \left| \mathscr{F}_j \right.\right) = \frac{1}{2}\left(ln\left(2\pi e\right) + ln\left(s_0\right) - ln\left(1 + s_0 \sum_{i=0}^{j-1} u_i^T B^T \left(\sigma\sigma^T\right)^{-1} B u_i\right)\right). \tag{6}$$

Remark 4 *Form (6) we see, that if the matrices $B^T\left(\sigma\sigma^T\right)^{-1}B$ is positively defined, then for any $u_i \neq col(0,0,...,0) \in \mathbb{R}^l$, $j = 0,1,...,N-1$ the entropy decreases when the horizon of control N increases. Additionally, the entropy decreases when the energy of control $\left\| u_j \right\|$, $j = 0,1,...,N-1$ (energy cost) is greater.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a growing function, which represents a penalty associated with the conditional entropy of random variable ξ. The value $f\left(H\left(\xi \left| \mathscr{F}_N \right.\right)\right)$ denotes a possible cost related to our ignorance of the system's behaviour (possible costs in future when the system will be controlled). Therefore, during the system control, we incur the following costs: direct costs of control $\sum_{i=0}^{N-1} u_i^T R u_i$, cost of not hitting to target $\left(y_N - a\right)^T Q \left(y_N - a\right)$ and the cost related to insufficient learning of parameters $f\left(H\left(\xi \left| \mathscr{F}_N \right.\right)\right)$. Below it has been assumed a penalty function as

$$f\left(x\right) = 1 - e^{-2x}.$$

From above, the penalty connected with ignorance of parameter ξ in system (1) is equal

$$f\left(H\left(\xi \left| \mathscr{F}_N \right.\right)\right) = 1 - \frac{1}{2\pi s_0}\left(1 + s_0 \sum_{i=0}^{N-1} u_i^T B^T \left(\sigma\sigma^T\right)^{-1} B u_i\right). \tag{7}$$

In many cases we bear the additional cost which are related not only to control. Sometimes during realization the aim, we acquire knowledge about the operation of the system additionally, which may be used in the future. Therefore this raises the question of how to control the system and also enrich our knowledge about it.

Below we consider the relation between costs of system control (1) and learning this system. For this purpose we modify the objective function. Let $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ be a coefficients of control cost, cost related to our ignorance of the system's behaviour and cost associated with the lack of hitting the target respectively. Thus the objective function can be presented as

$$J^N_{\alpha\beta\gamma}(u) = E\left(\alpha \sum_{i=0}^{N-1} u_i^T R u_i + \beta \left(y_N - a\right)^T Q \left(y_N - a\right)\right) + \gamma f\left(H\left(\xi \,|\mathscr{F}_N\right)\right). \quad (8)$$

From (7) the total cost is equal

$$J^N_{\alpha\beta\gamma}(u) = E\left(\sum_{i=0}^{N-1} u_i^T R_{\alpha\gamma} u_i + \beta \left(y_N - a\right)^T Q \left(y_N - a\right)\right) + \frac{2\pi s_0 - 1}{2\pi s_0}, \quad (9)$$

where

$$R_{\alpha\gamma} = \alpha R - \frac{\gamma}{2\pi} B^T \left(\sigma\sigma^T\right)^{-1} B. \quad (10)$$

The last component in equation (9) does not depend on the control. The matrix $R_{\alpha\gamma}$ in (10) denotes a compromise between costs connected directly with control and profits related to knowledge about system obtained during the control. The problem of optimal control consists in minimising the total costs. To determine the optimal control of system (1) we should solve the task

$$\inf_{u\in U} J^N_{\alpha\beta\gamma}(u), \quad (11)$$

i.e. it is necessary to find admissible control $u^* = \left(u_0^*, ..., u_{N-1}^*\right)$ for which the minimum is reached.

3 Optimal linear quadratic control relative to the cost of learning

In this section it will be presented the solution of task (11). To determine the optimal control for the system (1) the dynamic programming will be used. First we define the Bellman functions (see e.g. Bellman (1961), Fleming, Rishel (1975)). At the moment N the value of the Bellman function is equal

$$V_N^N(y_N) = \beta \left(y_N - a\right)^T Q \left(y_N - a\right) + \frac{2\pi s_0 - 1}{2\pi s_0}, \quad (12)$$

but at the moments $i = 0, 1, 2, ..., N-1$ it is defined as

$$V_i^N(y_i) = \min_{u_i} u_i^T R_{\alpha\gamma} u_i + E\left(V_{i+1}^N\left(y_{i+1}\right)\middle|\,\mathscr{F}_i\right). \quad (13)$$

From above the total cost of control with learning is equal

$$\inf_{u\in U} J^N_{\alpha\beta\gamma}(u) = V^N_0(y_0). \tag{14}$$

The theorem below presents the method of determining the optimal control.

Theorem 5 *Let* $\left\{\Psi_{j+1}(\xi)\right\}_{0\le j\le N}$ *be a sequence of matrices, which satisfy the equation*

$$\begin{aligned}\Psi_j(\xi) &= E\left(\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right) - E\left(\xi\Psi^T_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)B \\ &\times \left(R_{\alpha\gamma}+B^TE\left(\xi^2\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)B\right)^{-1}B^TE\left(\xi\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)\end{aligned} \tag{15}$$

with initial condition $\Psi_N(\xi)=\beta Q$. *If the matrices* $R_{\alpha\gamma}\in\mathbb{R}^{l\times l}$*and* $\Psi_j(\xi)\in\mathbb{R}^{n\times n}$ *are positively defined, then the solution of task (11) for stochastic system (1) is:*

- *optimal control* u^*_j *at the times* $0\le j\le N-1$ *is equal*

$$u^*_j = -\left(R_{\alpha\gamma}+B^TE\left(\xi^2\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)B\right)^{-1}B^TE\left(\xi\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)(y_j-a), \tag{16}$$

- *values of the Bellman function at times* $j=0,...,N-1$ *are calculated as*

$$V^N_j(y) = (y-a)^T\Psi_j(\xi)(y-a)+\varphi^N_j, \tag{17}$$

where

$$\varphi^N_j = \varphi^N_{j+1} + tr\left(\sigma^TE\left(\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)\sigma\right) \tag{18}$$

and $\varphi^N_N = \frac{2\pi s_0-1}{2\pi s_0}$.

Proof. At time N the Bellman function (12) may be presented in form (17), where $\Psi_N(\xi)=\beta Q$. Let us assume that the equation (17) is true for any $0<j+1\le N$ and next we check the truthfulness of this formula for the moment j. From (13) the value of Bellman function can be presented as

$$\begin{aligned}V^N_j(y_j) = \inf_{u_j}\Big\{&u^T_jR_{\alpha\gamma}u_j+\varphi^N_{j+1} \\ &+E\left((y_{j+1}-a)^T\Psi_{j+1}(\xi)(y_{j+1}-a)\Big|\mathscr{F}_j\right)\Big\}.\end{aligned} \tag{19}$$

and from the state equation (1) we have

$$\begin{aligned}V^N_j(y_j) = \inf_{u_j}\Big\{&u^T_jR_{\alpha\gamma}u_j+\varphi^N_{j+1} \\ &+E\left((y_j+\xi Bu_j+\sigma w_{j+1}-a)^T\Psi_{j+1}(\xi)(y_j-Bu_j+\sigma w_{j+1}-a)\Big|\mathscr{F}_j\right)\Big\}.\end{aligned}$$

Using the properties of conditional expectation value we obtain

$$V_j^N(y_j) = \inf_{u_j}\left\{u_j^T\left(R_{\alpha\gamma}+B^TE\left(\xi^2\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)B\right)u_j+\varphi_{j+1}^N \right.$$
$$+tr\left(\sigma^TE\left(\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)\sigma\right)+2u_j^TB^TE\left(\xi\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)(y_j-a)$$
$$\left.+(y_j-a)^TE\left(\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)(y_j-a)\right\}. \tag{20}$$

Thus the optimal control at the moment j is equal

$$u_j^* = -\left(R_{\alpha\gamma}+B^TE\left(\xi^2\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)B\right)^{-1}B^TE\left(\xi\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)(y_j-a). \tag{21}$$

Substituting the optimal control (21) into (20) that the value of the Bellman function $V_j^N(y_j)$ at time $0\leq j\leq N-1$ can be presented as (17), where $\Psi_j(\xi)$ and φ_j^N satisfy the equations (15) and (18) respectively. ■

Remark 6 *For the system (1) with unknown parameter ξ the optimal control (16) at time $0\leq j\leq N-1$ and the Bellman function $V_j^N(y_j)$ given by (17) depend both on current state y_j and previous states $y_0,...,y_{j-1}$.*

Remark 7 *At the moments $0\leq j\leq N-2$*

$$E\left(\Psi_{j+1}(\xi)\left|\mathscr{F}_j\right.\right)\neq\Psi_{j+1}\left(E\left(\xi\left|\mathscr{F}_j\right.\right)\right).$$

Replacing $\Psi_{j+1}(\xi)$ by $\Psi_{j+1}\left(E\left(\xi\left|\mathscr{F}_j\right.\right)\right)$ in formula (16) we obtain the self-tuning control, see e.g. Banek, Kozłowski (2005).

Remark 8 *For the system (1) with known parameter ξ (we have no cost related to ignorance of behaviour of system, thus $\gamma=0$) the optimal control is equal*

$$u_j^* = -\xi\left(R_\alpha+\xi^2B^T\Psi_{j+1}B\right)^{-1}B^T\Psi_{j+1}(y_j-a), \tag{22}$$

where $R_\alpha=\alpha R$ and matrices sequence $\left\{\Psi_{j+1}(\xi)\right\}_{0\leq j\leq N}$ satisfies the equation

$$\Psi_j = \Psi_{j+1}-\xi^2\Psi_{j+1}^TB\left(R_\alpha+\xi^2B^T\Psi_{j+1}B\right)^{-1}B^T\Psi_{j+1} \tag{23}$$

with initial condition $\Psi_N=\beta Q$. The Bellman function value at times $0\leq j\leq N$ is equal

$$V_j^N(y) = (y-a)^T\Psi_j(y-a)+\varphi_j^N, \tag{24}$$

where $\varphi_j^N=\varphi_{j+1}^N+tr\left(\sigma^T\Psi_{j+1}\sigma\right)$ and $\varphi_N^N=0$.

4 Conclusion

In this article the optimal control problem of discrete time stochastic linear system for fixed horizon was presented. The objective function is a sum of costs related to controls of system, heredity (costs due to lack of hitting the target) and learning of unknown parameter. To solve the task the dynamic programming was used, which is associated with classical concept of adaptation. This concept is based on tuning the controller by using the self-learning idea. In addition, the similarities and differences of control for the stochastic systems with known and unknown parameter were presented.

References

Aoki M. (1967), *Optimization of Stochastic Systems*, Academic Press.

Banek T., Kozłowski E., (2003a), *Extension of the portfolio models with an aspect of purchase of information*, Bulletin of the Polish Academy of Sciences. Technical Sciences, vol. 51(3), 227–243.

Banek T., Kozłowski E., (2003b), *On an operational model of single investment selection with information cost*, Control and Cybernetics, vol. 32(1), 87–102.

Banek T., Kowalik P., Kozłowski E., (1999), *Portfolio selection model with information cost*, Control and Cybernetics, vol. 28(1), 89–99.

Banek T., Kozłowski E., (2006), *Adaptive control of system entropy*, Control and Cybernetics, vol. 35(2), 279–289.

Banek T., Kozłowski E., (2005), *Active and passive learning in control processes application of the entropy concept*, Systems Sciences, vol. 31(2), 29–44.

Banek T., Kozłowski E., (2006), *Self-learning jako zadanie sterowania dualnego*, [in:] *Systemowo-komputerowe wspomaganie zarządzania wiedzą*, ed. by Kulikowski R., Bubnicki Z., Kacprzyk J., Exit, Warszawa, 157–236.

Banek T., Kozłowski E., (2011), *Active learning in discrete time stochastic systems*, [in:] *Knowledge-Based Intelligent System Advancements: Systemic and Cybernetic Approaches*, ed. by Józefczyk J., Orski D., Information Science References, New York, 350–371.

Banek T., Kozłowski E., (2014), *Deterministic versus stochastic in the free horizon LQC problem*, [in:] *Aktualne problemy automatyki i robotyki*, ed. by Malinowski K., Józefczyk J., Świątek J., Exit, Warszawa, 94–101.

Bellman R., (1961), *Adaptive Control Processes*, Princeton.

Fleming W. H., Rishel R., (1975), *Deterministic and Stochastic Optimal Control*, Springer-Verlag, Berlin.

Kozłowski E., (2018), *Optymalne sterowanie dyskretnymi systemami stochastycznymi*, Politechnika Lubelska, Lublin.

Kozłowski E., (2010), *The linear quadratic stochastic optimal control problem with random horizon at finite number of events independent of state system*, Systems Science, vol. 36(3), 5–11.

Kozłowski E., (2013), *Stabilization of linear system in random horizon via control*, Control and Cybernetics, vol. 42(2), 527–541.

Kozłowski E., (2011), *Identification of linear system in random time*, International Journal of Computer and Information Technology, vol. 1(2), 103–108.

Liptser R.Sh., Shiryaev A.N., (1978), *Statistics of Stochastic Processes*, Springer-Verlag, New York.

Saridis G.N., (1995), *Stochastic Processes, Estimation and Control: the Entropy Approach*, John Wiley and Sons.

Zabczyk J., (1996), *Chance and decision*, Scuola Normale Superiore, Pisa.

Tadeusz Banek

Two algorithms for computing the spherical center of n points

Keywords: optimal route, linear quadratic control, navigation, landmark

Abstract

We offer two approaches for computing the spherical center of n points. In first we use geometrical properties of a sphere $\mathbb{S}^2$ and we find a simple algorithm giving the approximate solution. In second we consider this problem as a typical linear programming problem.

1 Introduction

We consider the following problem

Problem 1 *Given points P_1,...,P_n on a unit hemisphere S^2 with a center in origin of $\mathbb{R}^3$ we have to find a point Q on S^2, such that the total distance*

$$d(Q;P_1,...,P_n) \triangleq \sum_{i=1}^{n} \rho^l(Q,P_i) \to \min, \tag{1}$$

where $\rho(Q,P_i)$ is a spherical distance (angle) between points Q and P_i, and $l > 0$.

(1) may be viewed as a simplified model of many problems arising in practice. In marine and aircraft transport operations for instance we are looking for a place where a total (spherical) distance to given ports is minimal. This place is a good candidate for a new central port which has to be build for ships or planes. This model can be easily generalized. If, for instance, some ports are visited more often then others and the resulting costs are higher then the total distance can be generalized

$$d(Q;P_1,...,P_n) \triangleq \sum_{i=1}^{n} w_i \rho^l(Q,P_i) \tag{2}$$

where $w_i > 0$ are the weights taking into account the transportation cost per angle multiplied by the visits frequency rate, etc. We propose two approaches which

correspond to the power $l=2$ and $l=1$ respectively. First is inspired by geometric observations and direct Second uses a linear programming procedure. We begin with geometric constructions which allow to transfer this spherical problem into a "flat" problem.

2 Geometric approach

2.1 From sphere to tangent plane

Let $\mathbf{T}_Q$ be a tangent plane at fixed point $Q \in S^2$. We define a mapping

$$S^2 \ni P \longmapsto f_Q(P) = Q + \rho^l(Q,P)\frac{P - \langle P,Q\rangle Q}{\|P - \langle P,Q\rangle Q\|} \in \mathbf{T}_Q, \tag{3}$$

which transfers points from S^2 to $\mathbf{T}_Q$. It is easy to check that $f_Q(P) \in \mathbf{T}_Q$, and $\|f_Q(P) - Q\| = \rho^l(Q,P)$, $\rho(Q,P) = \measuredangle(Q,P)$. By using this mapping we can transfer all points P_i to $\mathbf{T}_Q$ – a tangent plane at fixed, arbitrary chosen point $Q \in S^2$ getting

$$R_i = f_Q(P_i) = Q + \rho^l(Q,P_i)\frac{P_i - \langle P_i,Q\rangle Q}{\|P_i - \langle P_i,Q\rangle Q\|}. \tag{4}$$

Since $P_i - \langle P_i,Q\rangle Q$ lies on the plane spanned by Q and P_i, then $R_i \in \mathbf{T}_Q \cap span(Q,P_i)$ in the distance $\rho^l(Q,P_i)$ from Q. This construction can be easily visualized by taking a tangent plane at P_i and rolling it next to the tangent point at Q along geodetic $P_i \frown Q$. The traveling curve of P_i is known as evolventa

2.2 From tangent plane to sphere

Together with f_Q we shall need other mapping g_Q – an inverse to f_Q, defined by

$$B\left(Q, \left[\frac{\pi}{2}\right]^l\right) \ni R \longmapsto g_Q(R)$$

where

$$g_Q(R) = \left(\cos\|R-Q\| - \frac{\sin\|R-Q\|}{\sqrt{\|R\|^2-1}}\right)Q + \frac{\sin\|R-Q\|}{\sqrt{\|R\|^2-1}}R \in S^2, \tag{5}$$

and where $B(Q,r) = \{\mathbf{T}_Q \ni P; \|P-Q\| < r, r>0\}$, which transfers the points from $\mathbf{T}_Q$ to S^2. It is easy to check that $g_Q(R) \in S^2$. By using this mapping we can transfer points from $\mathbf{T}_Q$ to S^2 such that $\measuredangle(Q,g_Q(R)) = \rho(Q,g_Q(R))$, $\|R-Q\| = \rho^l(Q,g_Q(R))$. Similarly, this construction can be easily visualized by taking a tangent plane at Q and rolling it next to the tangent point at $g_Q(R)$ along geodetic $Q \frown g_Q(R)$. The point R will take the position $g_Q(R)$ as the result of this rotation and evolventa is the traveling curve of R.

2.3 Mean vectors

Given vectors P_1,...,P_n in $\mathbb{R}^m$, we set

$$\overline{P} = \frac{1}{n}\sum_{i=1}^{n} P_i \tag{6}$$

their mean value vector. It is well known that $\overline{P}$ solves the least squares problem, i.e.,

$$\sum_{i=1}^{n} \left\|\overline{P} - P_i\right\|^2 \leq \sum_{i=1}^{n} \|P - P_i\|^2 \tag{7}$$

for any $P \in \mathbb{R}^m$. Indeed, from

$$0 = \nabla_P \sum_{i=1}^{n} \|P - P_i\|^2 = 2\sum_{i=1}^{n} [P - P_i] \tag{8}$$

we conclude that the best P is $\overline{P}$.

2.4 Optimality conditions for l=2

It is easy to see that for the problem (1) with $l = 2$, the condition

$$\sum_{i=1}^{n} \rho^l (Q_\star, P_i) \frac{P_i - \langle P_i, Q_\star\rangle Q_\star}{\|P_i - \langle P_i, Q_\star\rangle Q_\star\|} = 0 \tag{9}$$

holds in the optimal point $Q_\star$.

2.5 The algorithm

Because is not likely to find a simple analytic formula which solves (1), we are looking for a dynamic way of reaching the best point $Q^\star$.

2.5.1 Step one

We begin with an arbitrary point $Q_0 \in S^2$. Let's map the points P_i on $\mathbf{T}_{Q_0}$ getting the points R_i^0 as is described in (3), i.e.,

$$R_i^0 = f_{Q_0}(P_i) = Q_0 + \rho^l (Q_0, P_i) \frac{P_i - \langle P_i, Q_0\rangle Q_0}{\|P_i - \langle P_i, Q_0\rangle Q_0\|}. \tag{10}$$

2.5.2 Step two

Having the points $R_i^0 \in \mathbf{T}_{Q_0}$ we compute their mean

$$\overline{R^0} = \frac{1}{n}\sum_{i=1}^{n} R_i^0. \tag{11}$$

Since (11) is a convex combination, hence $\overline{R^0} \in \mathbf{T}_{Q_0}$ is the best point for the 'flat' least total distance minimization problem.

2.5.3 Step three

Let's map the point $\overline{R^0}$ on the sphere S^2 by using (5). We get

$$Q_1 = g_{Q_0}\left(\overline{R^0}\right)$$

where

$$g_{Q_0}\left(\overline{R^0}\right) = \left(\cos\left\|\overline{R^0} - Q_0\right\| - \frac{\sin\left\|\overline{R^0} - Q_0\right\|}{\sqrt{\left\|\overline{R^0}\right\|^2 - 1}}\right) Q_0 + \frac{\sin\left\|\overline{R^0} - Q_0\right\|}{\sqrt{\left\|\overline{R^0}\right\|^2 - 1}}\overline{R^0} \tag{12}$$

as the first approximation of $Q_\star$.

2.5.4 Step four

Back to the step one, substituting Q^1 in place Q^0.

2.6 General

Generally, having Q_k we compute

$$R_i^k = f_{Q_k}(P_i) = Q_k + \rho^l(Q_k, P_i)\frac{P_i - \langle P_i, Q_k\rangle Q_k}{\|P_i - \langle P_i, Q_k\rangle Q_k\|}, \tag{13}$$

$$\overline{R^k} = \frac{1}{n}\sum_{i=1}^{n} R_i^k, \tag{14}$$

$$Q_{k+1} = g_{Q_k}\left(\overline{R^k}\right)$$

where

$$g_{Q_k}\left(\overline{R^k}\right) = \left(\cos\left\|\overline{R^k} - Q_k\right\| - \frac{\sin\left\|\overline{R^k} - Q_k\right\|}{\sqrt{\left\|\overline{R^k}\right\|^2 - 1}}\right) Q_k + \frac{\sin\left\|\overline{R^k} - Q_k\right\|}{\sqrt{\left\|\overline{R^k}\right\|^2 - 1}}\overline{R^k}. \tag{15}$$

2.7 Convergence

Define two mappings $S^2 \ni Q \longmapsto \phi(Q) \in \mathbf{Q}^\perp$, $S^2 \times \mathbf{Q}^\perp \ni (Q, W) \longmapsto \psi(Q, W) \in S^2$, by the formulae

$$\phi(Q) = \frac{1}{n}\sum_{i=1}^{n} \rho^l(Q, P_i)\frac{P_i - \langle P_i, Q\rangle Q}{\|P_i - \langle P_i, Q\rangle Q\|}, \tag{16}$$

$$\psi(Q, W) = Q\cos\|W\| + \frac{W}{\|W\|}\sin\|W\|, \tag{17}$$

and where $\mathbf{Q}^{\perp}$ is the orthogonal complement in $\mathbb{R}^3$ of the line spanned by Q and note, that (13)(14)(15) implies that the sequence Q_k, $k = 0, 1, ...$ described in the Algorithm is generated by the equation

$$Q_{k+1} = \omega(Q_k). \tag{18}$$

where

$$\omega(Q) = \psi(Q, \phi(Q)),$$

is given explicitly by the formula

$$\begin{aligned} \omega(Q) = Q\cos\left\| \frac{1}{n}\sum_{i=1}^{n} \rho^l(Q,P_i)\, \frac{P_i - \langle P_i, Q\rangle Q}{\|P_i - \langle P_i, Q\rangle Q\|} \right\| \\ + \frac{\sum_{i=1}^{n} \rho^l(Q,P_i)\, \frac{P_i - \langle P_i, Q\rangle Q}{\|P_i - \langle P_i, Q\rangle Q\|}}{\left\| \sum_{i=1}^{n} \rho^l(Q,P_i)\, \frac{P_i - \langle P_i, Q\rangle Q}{\|P_i - \langle P_i, Q\rangle Q\|} \right\|} \sin\left\| \frac{1}{n}\sum_{i=1}^{n} \rho^l(Q,P_i)\, \frac{P_i - \langle P_i, Q\rangle Q}{\|P_i - \langle P_i, Q\rangle Q\|} \right\|. \end{aligned} \tag{19}$$

To prove that Q_k converges to $Q_\star$, such that

$$d(Q_\star; P_1, ..., P_n) \leq d(Q; P_1, ..., P_n),\ Q \in S^2,$$

it is enough to show that the mapping $Q \longrightarrow \omega(Q)$ in (18) is a contraction. Indeed, by the famous Banach's fixed point theorem the contraction of $\omega(\cdot)$ implies convergence of Q_k to the point $Q^\star$, such that

$$Q^\star = \omega(Q^\star) = \psi(Q^\star, \phi(Q^\star)), \tag{20}$$

and from (16) and (17) we know that is possible only when the optimality condition (9), $\phi(Q^\star) = 0$, is satisfied. This implies that $Q^\star = Q_\star$. In consequence we have the following useful

Conclusion 2 *The algorithm can reach the optimal $Q_\star$ with arbitrary small error.*

Proof. It is easy to see that (19) is a contraction for all $Q \in S^2$, except for one point $Q = Q_\star$. Any fixed approximation error $\varepsilon = \|Q - Q_\star\|$ corresponds to a disc in S^2 with a center $Q_\star$ and radius ε. Outside the disc the constant of contraction bigger than one by a positive number, hence the algorithm is convergent to some point of the disc. ■

3 Linear programming approach

In the particular case $l = 1$, we shall show how the problem (1) can be reduced to the classical LP problem.

Let introduce some notations $J = \mathrm{col}\,(1,...,1) \in \mathbb{R}^n$, $e_i = \mathrm{col}\,(0,,,0,1,0,...0)$. Let us denote by α_{ij} the angle between P_i and P_j, by x_i the angle between Q and P_i, and $x = \mathrm{col}\,(x_1,...,x_n)$. Note, that we always have

$$\langle e_i + e_j, x\rangle = x_i + x_j \geq \alpha_{ij}$$

for any $i, j = 1,...,n$, $i \neq j$. Since for $l = 1$,

$$d\,(Q;P_1,...,P_n) = \langle J,x\rangle \tag{21}$$

thus the problem (1) can be described in the form

$$\min\left\{\langle J,x\rangle\,;\langle e_i + e_j,x\rangle \geq \alpha_{ij}, x_i \geq 0, \text{ for } i,j = 1,...,n,\, i \neq j\right\}$$

In order to express this problem in more familiar form of the linear programming language, i.e., a triple; c, A, b, note that

$$c = x_1 + ... + x_n$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} \alpha_{11} \\ . \\ \alpha_{1n} \\ \alpha_{23} \\ . \\ \alpha_{2n} \\ . \\ . \\ \alpha_{n-1,n} \end{bmatrix}$$

and the problem is

$$c = x_1 + ... + x_n \to \min$$

under the conditions

$$\begin{aligned} Ax &\geq b \\ x &\geq 0. \end{aligned}$$

Remark 3 *It is worth nothing that the LP algorithm may be used in the case $l = 2$ as well. Indeed in this case*

$$d\,(Q;P_1,...,P_n) = \|x\|^2$$

thus the problem (1) can be described in the form

$$\min\left\{\|x\|^2;\left\langle e_i+e_j,x\right\rangle\geq\alpha_{ij},x_i\geq 0,\ \textit{for } i,j=1,...,n,\ i\neq j\right\} \tag{22}$$

what means that we are looking for the point of minimal norm in the set $Ax\geq b$, $x\geq 0$. However, this point appears to be solution of the following problem

$$\min\left\{\min\left\{\langle y,x\rangle;\left\langle e_i+e_j,x\right\rangle\geq\alpha_{ij},x_i\geq 0,\ \textit{for } i,j=1,...,n,\ i\neq j\right\};y\in\mathbb{R}^n\right\},$$

i.e., finding solutions $x(y)$ of the LP problems corresponding to fixed $y\in\mathbb{R}^n$, and vary y over $\mathbb{R}^n$ next, we may find the solution of (22) by making the suitable selection among $x(y)$.

Remark 4 *For the problem (2) it is enough to modify the criterion (21) as follows. Take*

$$d(Q;P_1,...,P_n)=\langle w,x\rangle$$

instead of $d(Q;P_1,...,P_n)=\langle J,x\rangle$ and the problem (2) reads now: given $w\in\mathbb{R}^n$, find

$$\min\left\{\langle w,x\rangle;\left\langle e_i+e_j,x\right\rangle\geq\alpha_{ij},x_i\geq 0,\ \textit{for } i,j=1,...,n,\ i\neq j\right\}.$$

References

Dantzig G., (1963), *Linear Programming and Extensions*, Princeton University Press.

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A practical approach to point processes

Keywords: Poisson point process, Cox process, point patterns, Wasserstein distance, Statistica, Mathematica, R, EasyFit

Abstract

The goal of this article is to provide the reader with practical tools for investigating point processes. These include: acceptance-rejection procedure, independence hypothesis testing, point pattern analysis. To this end we use Statistica 13.2, Mathematica, R, EasyFit and Microsoft Excel.

1 Introduction

In recent years, the theory of point processes has been successfully developed (e.g. introducing Ripley's K-function, etc) and applied in many disciplines, such as forestry, zoology, geography or medicine, see Baddeley, Rubak and Turner (2016), Daley and Vere-Jones (2003), Diggle (2015), Streit (2010) and references therein. Additionally there is much more tools available now for point patterns analyzing and statistical computing than in previous years. The R project (www.r-project.org) is an example of such a tool. In this article we present various statistical tests and techniques which may be used in frontline research in agricultural or physical sciences. We focus our attention on Poisson process, Cox process, independence and CSR testing.

2 Definition and simulation

Just a few definitions at the beginning. We assume throughout that $(S, \mathscr{B}(S))$ is a measurable space. In what follows, we may think that S is a subset of $\mathbb{R}^d$. The Lebesgue measure of $A \in \mathscr{B}(\mathbb{R}^d)$ is denoted here by $|A|$. A Borel measure on S is a nonnegative measure that is finite on compact sets. A point process or a random subset is a measurable mapping Π from some underlying probability space into the

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set of all countable subsets of S. For fixed A, the random variable $N(A) := |\Pi \cap A|$ counts the points that "fall" into A. Let μ be a finite measure on S, i.e. $\mu(S) < \infty$, not necessarily a probability measure. A Poisson process on S with the mean measure μ is a random countable subset Π of S such that the following conditions hold

(i) for any $A \in \mathscr{B}(S)$, $N(A)$ has the Poisson distribution with parameter $\mu(A)$, i.e.
$$\mathbb{P}(N(A) = k) = \frac{\mu^k(A)}{k!} e^{-\mu(A)}, \quad k = 0, 1, 2, \ldots$$

(ii) for any $n \geqslant 2$ and any pairwise disjoint subsets $A_1, \ldots, A_n$ of S, the random variables $N(A_1), \ldots, N(A_n)$ are independent.

Thus we have
$$\mathbb{E}[N(A)] = D^2[N(A)] = \mu(A) \quad \left(= \int_A \lambda(x) dx \right), \tag{1}$$
if μ is absolutely continuous with respect to the Lebesgue measure. We also have
$$\begin{cases} \mathbb{E}[N(A)N(B)] = \mu(A)\mu(B) + \mu(A \cap B) \\ \operatorname{cov}(N(A), N(B)) = \mu(A \cap B), \end{cases} \tag{2}$$
for any $A, B \in \mathscr{B}(S)$, see Kingman (1993) or Daley and Vere-Jones (2003). The function $\lambda(x)$ in (1) is called the intensity of Π. If λ is constant then Π is termed as homogeneous process. Otherwise it is referred to as nonhomogeneous. A sample ξ of Π consists of two things: n is the number of points and $x_1, \ldots, x_n$ are points of S. We write $\xi = (n, \{x_1, \ldots, x_n\})$. Here is how we simulate a sample of a Poisson process.

Suppose that $\mu(S) = \int_S \lambda(x) dx > 0$. Then

(i) We choose $n \geq 0$ by sampling the discrete Poisson random variable with parameter $\mu(S)$, that is $\mathbb{P}(N = n) = e^{-\mu(S)} \mu^n(S)/n!$ If $n = 0$ then the realization is $(0, \emptyset)$. If $n > 0$ then we perform (ii).

(ii) The points $x_1, x_2, \ldots, x_n$ are obtained as iid samples of a random variable X with pdf given by
$$p_X(x) = \frac{\lambda(x)}{\mu(S)}, \quad x \in S. \tag{3}$$

The output is n and the vector $(x_1, \ldots, x_n)$, however the order of elements is irrelevant here, so the realization is $\xi = (n, \{x_1, \ldots, x_n\})$. If S is a continuous subset of $\mathbb{R}^d$ then the points $\{x_1, \ldots, x_n\}$ are distinct with probability one.

Next we use an acceptance-rejection procedure to generate iid samples of (3). Here we explain this procedure in detail.

1. First, choose any bounded pdf $g(x) > 0, x \in S$, from which iid samples of S can be generated by a known procedure. For example, it can be a uniform distribution on S. The importance function is another term for $g(x)$.

2. Calculate
$$M = \max_{x \in S} \frac{p_X(x)}{g(x)}$$
and note that then $p_X(x) \leq Mg(x)$, see Fig. 1.

3. Draw random $n \in \{0, 1, 2, \ldots\}$ from Poisson distribution with mean $\mu(S)$.

4. Draw random x with pdf g, compute $t = p_X(x)/Mg(x)$ and draw random sample u, uniformly distributed on $[0, 1]$.

5. Accept x, if $u \leq t$. Reject x, if $u > t$. Why do we do so? For fixed x we choose temporarily, name it, y-coordinate for x uniformly on $(0, Mg(x))$. This y is just $uMg(x)$, where u is from (4). Then we accept x if the point (x, y) lies under the graph of p_X, i.e. $uMg(x) \leq p_X(x)$ which is equivalent to $u \leq t$.

Stop when n points are accepted. Note that those "accepted" points (x_i, y_i) are uniformly distributed in the area under the graph of p_X, hence marginally $x_i's$ are iid samples of p_X, see again Fig 1.

We performed calculations for three different intensities, one realization per intensity, see Fig. 2. Since discussed procedure is straightforward and if one needs to simulate a sample of small size, Microsoft Excel is good for calculations. In other cases we recommend the R packages. We will use the R software later in this article.

Here are some details about intensities. Since $\lambda_1(x, y) = 2$, then the number of points in $[-2, 2] \times [-2, 2]$ has the distribution $Poiss(32)$. The generator of random numbers drawn 33 points in this case. Next, we have $\lambda_2(x, y) = 0.25e^{2x+y}$ and

$$0.25 \int_{-2}^{2} \int_{-2}^{2} e^{2x+y} dxdy \approx 49.48,$$

so we have 50 points in this sample (accepted from 1100 points). The last one, $\lambda_3(x, y)$ is proportional to the normal distribution:

$$\lambda_3(x, y) = 30e^{-2x^2-4y^2}, \quad (x, y) \in [-2, 2] \times [-2, 2].$$

Since

$$\int_{-2}^{2} \int_{-2}^{2} \lambda_3(x, y) dxdy \approx 33.3195,$$

then n was drawn according to $Poiss(33.3195)$ and we obtained $n = 35$. Those 35 points in the picture were accepted from about 500 uniformly distributed points on this square.

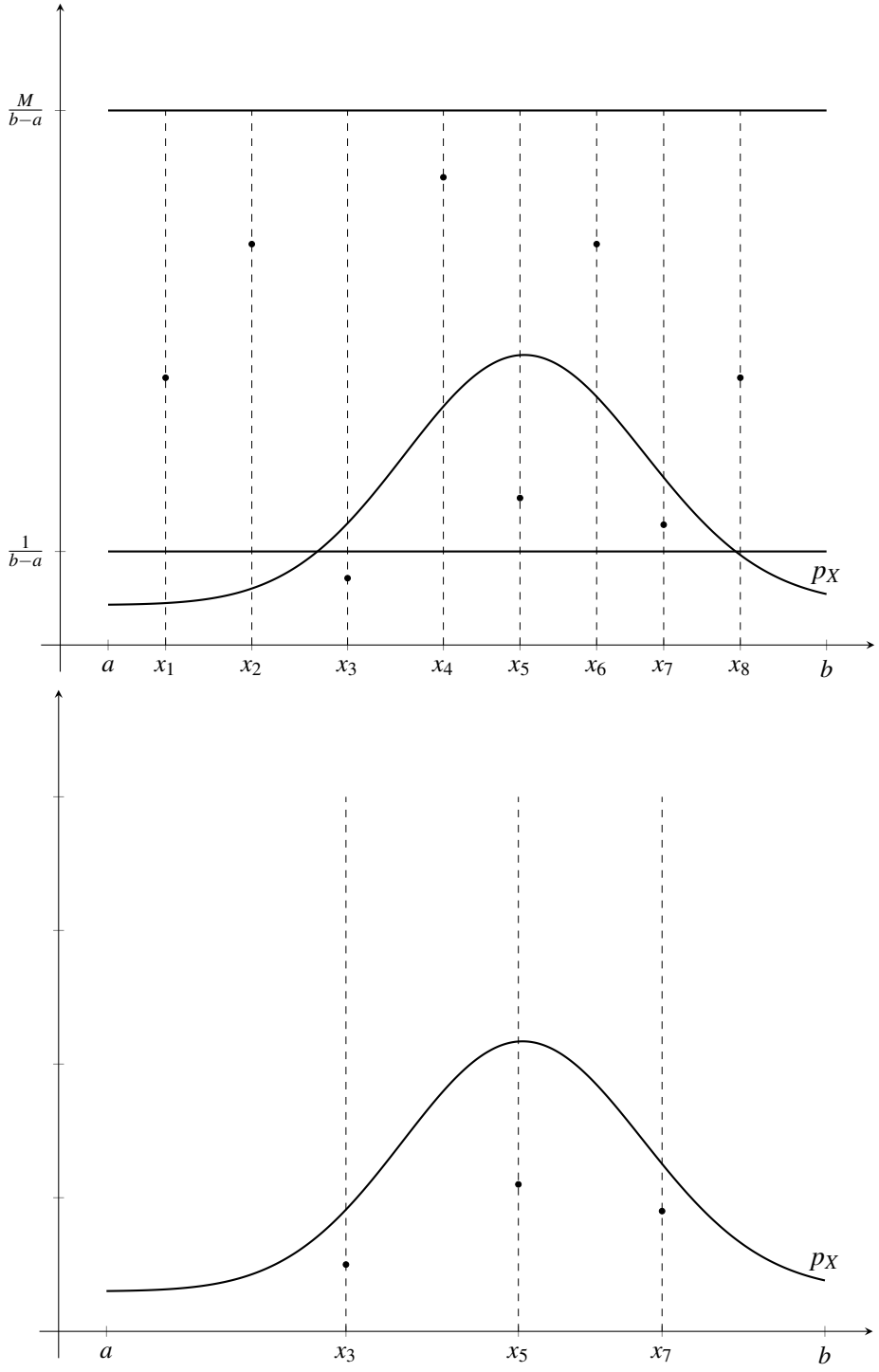

Figure 1. The acceptance-rejection procedure for $S=[a,b]$ and $g(x)=1/(b-a)$. Accepted points x_3,x_5,x_7 are iid samples of p_X

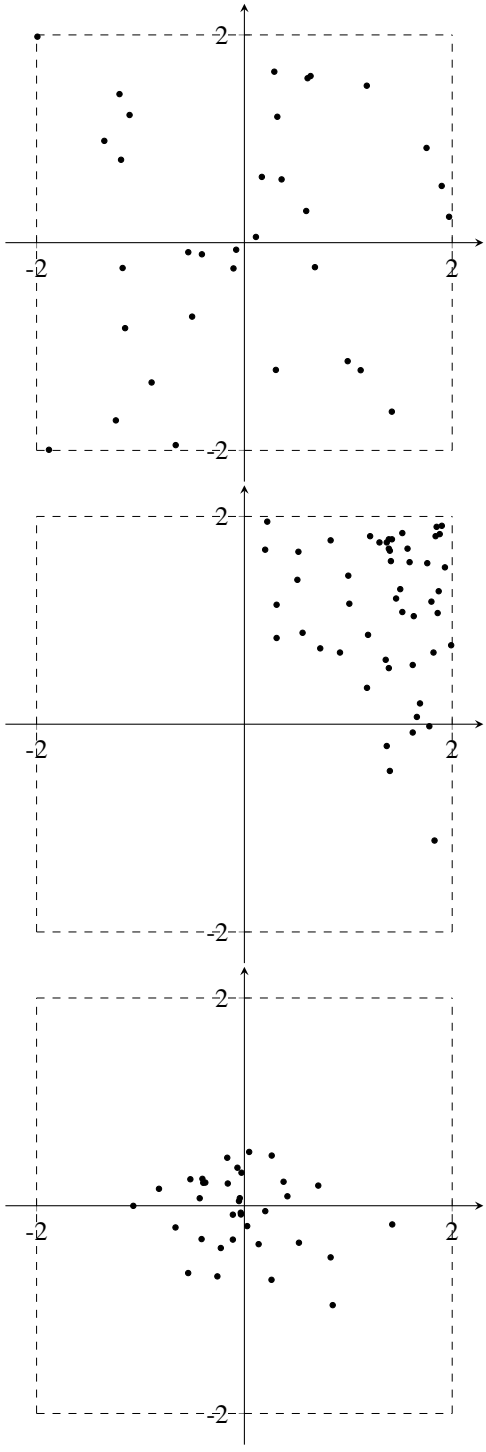

Figure 2. Realizations of a Poisson process on the square $[-2,2]\times[-2,2]$ with various intensity functions: $\lambda_1(x,y)=2$ (top), $\lambda_2(x,y)=0.25e^{2x+y}$ (middle), $\lambda_3(x,y)=30e^{-2x^2-4y^2}$ (bottom)

3 Nearest-neighbour distance and fitting distributions

It is well known that if Π is a Poisson process of constant rate λ on $(0,+\infty)$ and the points of Π be written as $0<X_1<X_2<\ldots$ then $Y_1=X_1$, $Y_2=X_2-X_1,\ldots$ are independent and each has the density $f(x)=\lambda e^{-\lambda x}$, for $x>0$. In other words: increments are independent and exponentially distributed. In this case realizations are usually plotted as integer valued step-functions. Incidentally, the distribution of X_n is that of gamma with parameters n,λ. An interested reader will find those facts in any book devoted to stochastic processes, e.g. Wentzell (1980) or Daley and Vere-Jones (2003).

As for higher dimensions, denote by X the distance of the point closest to the origin of a Poisson process in $\mathbb{R}^d$ with constant rate λ. Then it satisfies $\mathbb{P}(X>r)=\exp(-\lambda v_d(r))$, $r>0$, where $v_d(r)=r^d v_d(1)$ is the volume of a sphere of radius r in $\mathbb{R}^d$, see e.g. Chapter 2 of Daley and Vere-Jones (2003). Hence, the density of the distance of the nearest point is

$$f(r)=\lambda v_d(1) d r^{d-1} e^{-\lambda v_d(1) r^d}, \quad r>0,$$

and $d=1,2,\ldots$. From this we have

$$\begin{aligned}\mathbb{E}(X) &= \lambda v_d(1)\int_0^\infty r d r^{d-1} e^{-\lambda v_d(1) r^d} dr \\ &= \frac{1}{\sqrt[d]{v_d(1)}\sqrt[d]{\lambda}}\int_0^\infty t^{\frac{1}{d}} e^{-t} dt \\ &= \frac{\Gamma(1+\frac{1}{d})}{\sqrt[d]{v_d(1)}}\frac{1}{\sqrt[d]{\lambda}}, \quad d=1,2,3,\ldots\end{aligned}$$

Recall that

$$v_d(1)=\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2}+1)$$

and

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,$$

where $x>0$.

Sometimes one needs to investigate certain aspects of a point pattern (mean, distance, etc) that require to fit a distribution to the collected data, see e.g. Long et al. (2014). The next example is devoted to this problem. For a point process Π on the line and a realization $\{x_1,\ldots,x_n\}$ define

$$D=\frac{x_1+x_2+\ldots+x_n}{n}, \quad M=\max(x_1,\ldots,x_n).$$

Table 1. Pdf's and expectations of nearest-neighbour distance for a homogeneous Poisson process

dim, d	$\mathbb{P}(X > r)$	density, $f(r)$	$\mathbb{E}(X) = \int_0^\infty r f(r) dr$
1 ($\mathbb{R}_+$)	$e^{-r\lambda}$	$\lambda e^{-r\lambda}$	$\frac{1}{\lambda}$
1 ($\mathbb{R}$)	$e^{-2r\lambda}$	$2\lambda e^{-2r\lambda}$	$\frac{1}{2\lambda}$
2 ($\mathbb{R}^2$)	$e^{-\pi r^2\lambda}$	$2\pi r\lambda e^{-\pi r^2\lambda}$	$\frac{1}{2\sqrt{\lambda}}$
3 ($\mathbb{R}^3$)	$e^{-\frac{4}{3}\pi r^3\lambda}$	$4\pi r^2\lambda e^{-\frac{4}{3}\pi r^3\lambda}$	$\approx \frac{0.554}{\sqrt[3]{\lambda}}$
4 ($\mathbb{R}^4$)	$e^{-\frac{1}{2}\pi^2 r^4\lambda}$	$2\pi^2 r^3\lambda e^{-\frac{1}{2}\pi^2 r^4\lambda}$	$\approx \frac{0.608}{\sqrt[4]{\lambda}}$

Example 1 *We simulated 30 independent realizations of a nonhomogeneous Poisson process on an interval* $[0,10]$ *with intensity* $\lambda(x) = 10e^{-x}$, $x \in [0,10]$. *Histograms of D and M are presented in Fig. 3. We obtained* $\overline{D} = 0.9$, $s_D = 0.3$ *and* $\overline{M} = 2.41$, $s_M = 1.28$. *Furthermore* $D \in [0.41, 1.5]$ *and* $M \in [0.84, 6.21]$. *To study D and M, we used a chi-squared and the Kolmogorov-Smirnov goodness of fit tests. For D, the K-S test produced P-value 0.98 for the log-normal distribution with parameters* $\mu = 0.32$, $\sigma = 0.21$ *and* $\gamma = -0.52$, *see also Fig. 3. The density of the log-normal distribution is given by*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}(x-\gamma)} \exp\left(-\frac{1}{2\sigma^2}(\ln(x-\gamma)-\mu)^2\right),$$
$$x \in (\gamma, +\infty).$$

Here $\mu \in \mathbb{R}$, $\sigma > 0$ *are continuous parameters and* γ *is a location parameter. If X is a rv with this distribution then* $\mathbb{E}(X) = \gamma + e^{\mu+\frac{1}{2}\sigma^2}$ *and* $var(X) = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}$. *Thus the mean with fitted parameters equals 0.8878 and variance* $0,298$.

As for M, the best fit with P-value 0.87 is for log-logistic distribution with $\alpha = 3.28$, $\beta = 1.98$ *and* $\gamma = 0.14$, *see Fig. 3. The pdf of this distribution is*

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\left(1+\frac{(x-\gamma)^{\alpha}}{\beta}\right)^{-2}, \quad x \in (\gamma, +\infty),$$

where $\alpha > 0$ *(shape parameter),* $\beta > 0$ *(scale parameter) and the location parameter* $\gamma \in \mathbb{R}$. *We performed this analysis using EasyFit Professional, version 5.6.*

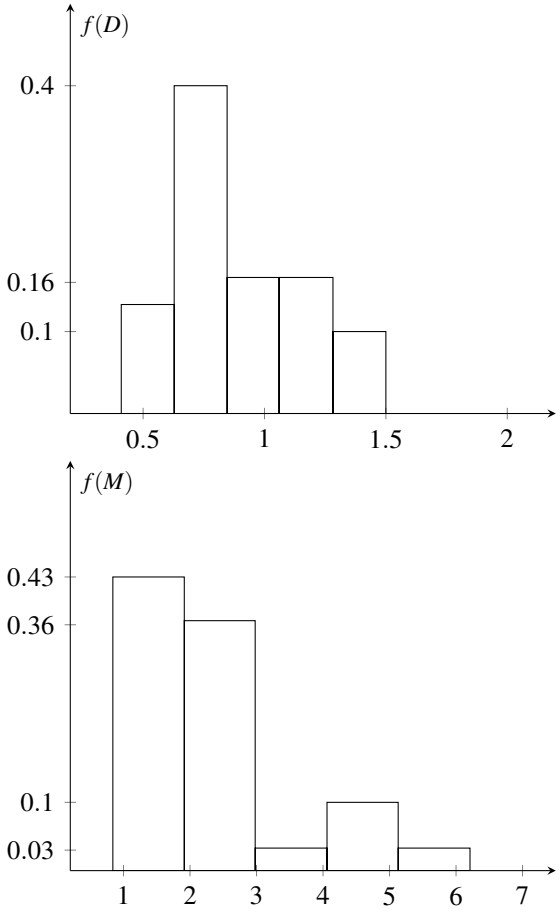

Figure 3. Histograms and fitted distributions of D (average distance) and M (maximum) for a Poisson process with intensity $\lambda(x) = 10e^{-x}$, $x \in [0,10]$

4 Testing Independence with Statistica and Wolfram

In the next section we investigate complete spatial randomness of a point pattern in a plane. It is worth to look at how we test for independence of random vectors before that. To this end we consider a couple of examples.

This is the first one. Although simplified, a typical situation in many physical phenomena looks like this one shown in Fig. 4. See e.g. Long et al. (2014) and references therein.

Namely, a black ball is dropped into the central box containing 30 smaller white balls as shown in Fig. 4. After the black ball hits the box, a number of white balls is thrown out the central box and some white balls fall into the right

box and some into the left one. Let X denote the number of white balls that are contained in the right box and Y in the left box. We want to test whether X and Y are statistically independent. We carried out 15 independent trials and here are the outcomes: (4,3), (3,3), (4,2), (3,5), (2,4), (5,4), (3,6), (2,4), (4,5), (4,3), (3,4), (4,4), (4,3), (5,3), (4,5). So we need to determine whether the vectors

$$\begin{cases} \vec{x} = (4,3,4,3,2,5,3,2,4,4,3,4,4,5,4), \\ \vec{y} = (3,3,2,5,4,4,6,4,5,3,4,4,3,3,5) \end{cases} \tag{4}$$

are statistically independent. Hence the null hypothesis H_0: X and Y are statistically independent. The alternative one is H_1: X and Y are associated (i.e. not independent). Note that in H_1 the relationship between X and Y is not specified.

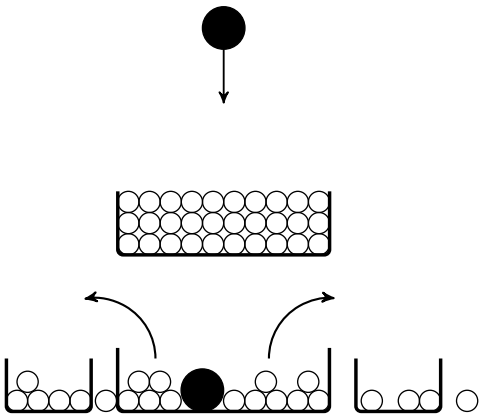

Figure 4. Here X denotes the number of balls in the right box, and Y in the left one. In this particular case $X = 3$ and $Y = 5$. Tests support the null hypothesis, i.e. X, Y are statistically independent

The most widely used is the Pearson's χ^2 test. The test statistic is given by

$$\chi^2 = \sum_{i=1}^{k} \sum_{j=1}^{r} \frac{(n_{ij} - \hat{n}_{ij})^2}{\hat{n}_{ij}} = \sum_{i=1}^{k} \sum_{j=1}^{r} \frac{\hat{n}_{ij}^2}{\hat{n}_{ij}} - n,$$

where n_{ij} ($\hat{n}_{ij}$) is the true (expected) number of observations of type i and j, where k is the number of types of observations of X and r the number of types of Y. Here n is the total number of observations. Considered statistic has asymptotic distribution χ^2 with $(r-1)(k-1)$ degrees of freedom (df). In our case, X and Y are integer-valued random variables. For vectors (4) we have $\chi^2 = 9.78$, df=12, with P-value 0.63. Since P-value is above 0.05, so there is not enough evidence to reject H_0 at that level, i.e. we support H_0.

More accurate tests are also available. Usually these are based on measures of association between rv's. For example, it may be Blomqvist's medial correlation

coefficient for two vectors, Hoeffding's independence test, etc. We carried out these tests using Statistica 13.2 and WolframAlpha (available online), see Table 2 below.

Table 2. Statistic values for sample vectors (4) support the hypothesis that X and Y are statistically independent

Statistica 13.2		
	Statistic	P-value
Pearsons's χ^2	9.78	0.63
Cramer's V	0.46	
Kendall's τ (b)	-0.25	
Kendall's τ (c)	-0.23	
Spearman's rank	-0.31	0.24
Wolfram (Mathematica)		
	Statistic	P-value
Blomqvist's β	-0.31	0.11
Goodman-Kruskal's γ	-0.34	0.34
Hoeffding's D	-0.04	0.83
Kendall's τ	-0.25	0.26
Spearman's rank	-0.31	0.24

Consider a second example. A random phenomenon produces a single point $P=(X,Y)$ inside the rectangle $[-2,2]\times[-1,1]$. We generated 15 independent realizations of this phenomenon, see Fig. 5, and the points are (coordinates are given to 1 decimal place): (1.1, 0.6), (-1, 0.8), (-0.2, -0.1), (1.2, -0.6), (1.6, 1.7), (0.2, 0.4), (-1.4, 0.5), (-1.8, -0.8), (1.5, 0.1), (1.7, -0.2), (0.6, 0.8), (-0.3, -0.6), (0, 0.3), (-0.6, 0.3), (0.7, 0). Again, we want to test if coordinates of P are independent.

Although given only to 1 decimal place (for obvious reasons), those are in fact real-valued vectors:

$$\begin{cases} \vec{x}= & (1.1,-1,-0.2,1.2,1.6,0.2,-1.4,-1.8,\\ & 1.5,1.7,0.6,-0.3,0,-0.6,0.7),\\ \vec{y}= & (0.6,0.8,-0.1,-0.6,0.7,0.4,0.5,-0.8,\\ & 0.1,-0.2,0.8,-0.6,0.3,0.3,0) \end{cases} \tag{5}$$

and before performing χ^2 test directly, we need to create a contingency table for $\vec{x},\vec{y}$. Namely, we divide the given rectangle into smaller subsets (called quadrats) and count the number of points in each quadrat, see Fig. 5. Then we carry out the χ^2 test based on the information contained in the contingency table. In our case $\vec{x}$, $\vec{y}$ are transformed to integer-valued vectors

$$\begin{cases} x'=(1,2,2,3,3,3,3,4,4,4,1,2,2,4,4),\\ y'=(1,1,1,1,1,1,1,1,1,1,2,2,2,2,2) \end{cases}$$

where the meaning of those integers is this: we divided $[-2,2]$ into 4 subintervals and numbered them from 1 to 4. The same with y-coordinate: we divided $[-1,1]$ into $[-1,0)$ (denoted by 1) and $[0,1]$ (denoted by 2). For x', y' we have $\chi^2 = 2.85$ with P-value 0.41, meaning that we do not reject the null hypothesis.

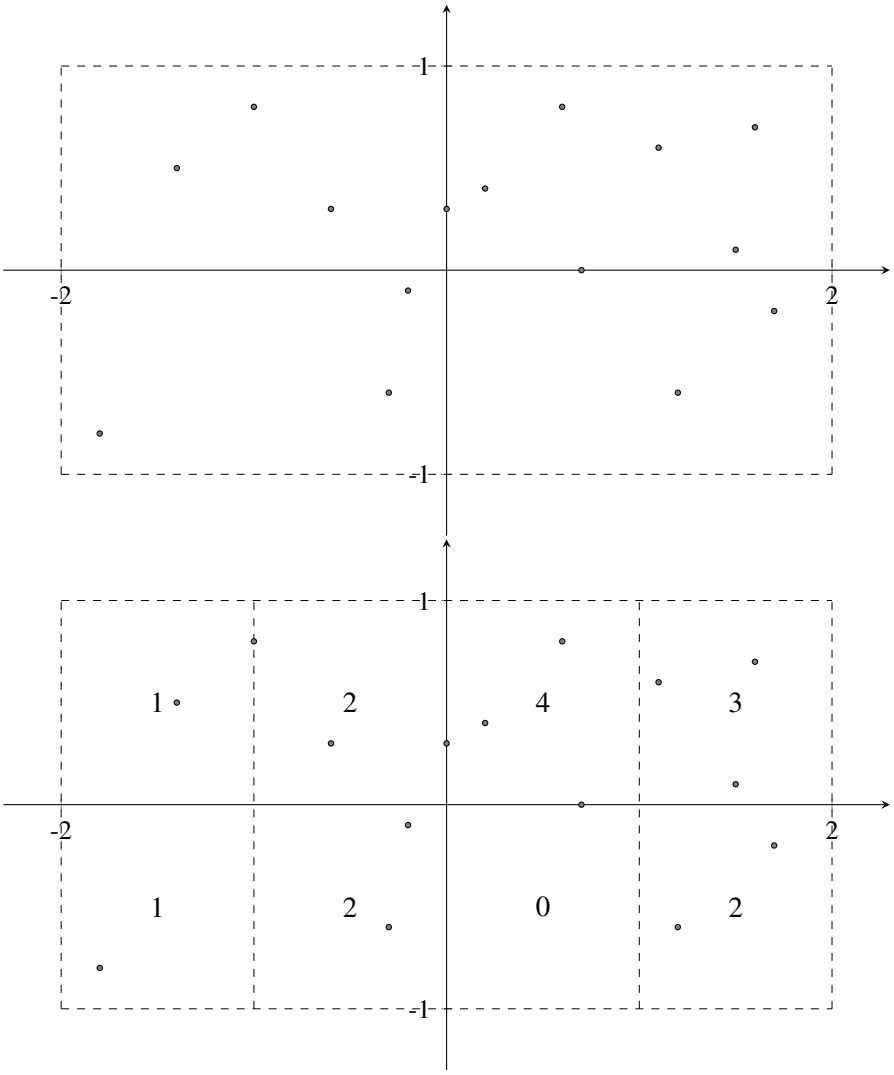

Figure 5. Independent realizations of certain random phenomenon (top, there are 15 dots) and quadrat counts (bottom)

And the last example. Namely, consider

$$\begin{cases} \vec{x} = (2,-1,3,3,4,-2,0,1,-1,5,-2,4,3,1,-1), \\ \vec{y} = (4,1,9,9,16,4,0,1,1,25,4,16,9,1,1). \end{cases} \tag{6}$$

For integer-valued vectors (6) we have $\chi^2 = 75$, df=35, with P-value 0.0001, so there is strong evidence against H_0. Hence we support H_1 which says that there is a relationship between $\vec{x}$ and $\vec{y}$. Obviously $\vec{y} = \vec{x}^2$, see also Table 3 below.

Table 3. Statistic values for obviously dependent vectors (6), i.e. $\vec{y} = \vec{x}^2$

Statistica 13.2		
	Statistic	P-value
Pearsons's χ^2	75.00	0.00
Cramer's V	1	
Kendall'a τ (b)	0.63	
Kendall'a τ (c)	0.61	
Spearman's rank	0.75	0.00
Wolfram (Mathematica)		
	Statistic	P-value
Blomqvist's β	0.80	0.00
Goodman-Kruskal's γ	0.65	0.00
Hoeffding's D	0.29	0.00
Kendall's τ	0.63	0.00
Spearman's rank	0.75	0.00

5 Complete spatial randomness with R

The purpose of this section is to test whether a given point pattern on the plane has a CSR property. We begin with definition.

We say that a point pattern has a complete spatial randomness (CSR) property or is simply CSR if it is a homogeneous Poisson point process in the plane. Thus if $N(A) = |\Pi \cap A|$, where $A \in \mathscr{B}(\mathbb{R}^2)$ with $|A| < \infty$, we have

(a) $N(A)$ has a Poisson distribution with mean $\lambda|A|$,

(b) assuming that there are n points in A, their locations are iid and uniformly distributed inside A,

(c) $N(A), N(B)$ are independent for disjoint A and B.

We generated 40 points in the rectangle (or window) $W := [-10,10] \times [-5,5]$. These are (again, coordinates are given with accuracy of one decimal place): (1.1,2), (8.3,3.1), (5,2.8), (2.2,3), (4,4.5), (9,1.5), (0.4,4.9), (6,4), (7.1,2.6), (8.7,0.4), (3.5,

3.9), (9.5,2), (-2,4), (-6.5,1), (-8,0.5), (-3.1,2), (-1,4.5), (-4,2.5), (-0.3,4.8), (-2.5,-3.7), (-6.1,-4.2), (-0.1,-3.7), (-7,-0.4), (-5.6,-2.8), (-9.7,-1.5), (-3.5,-2.1), (-0.8,-3), (0,-3.3), (1,-4.2), (2.7,-2), (4.1,-0.3), (5,-1.7), (5.4,0.5), (6.1,-1.1), (6.9,-4.2), (7.8,-3.5), (8.5,-0.1), (9,-2), (9.2,-4.5), (9.6,2.6). The main package used here in this analysis is `spatstat` of R. We have at least three tests at our disposal:

(1) χ^2 test;

(2) Kolmogorov-Smirnov test of CSR;

(3) Ripley's K-function;

The first one, χ^2 test is based on quadrat counts. Namely, as in section 4 we divide the window W into quadrats $A_1,\ldots,A_m$ and count the numbers $n_1,\ldots,n_m$ of points in each quadrat. If the null hypothesis is true, the n_i are realizations of independent Poisson random variables with expected values $\lambda|A_i|$, where λ is the unknown intensity. We perform this test by `quadrat.test(P,nx=4,ny=2)`, where `P` is our point patter, i.e. the set of 40 points in W. The result is $\chi^2 = 4.4$, df $= 7$ and P-value $= 0.5346$, see also Fig. 6. This supports the null hypothesis that the point pattern is CSR.

It is also possible to conduct a one-sided test, and to compute the P-value using Monte Carlo simulation instead of the χ^2 approximation.
By typing

```
> quadrat.test(P, 8, alternative="regular", method="MonteCarlo")
```

we get the results

```
Conditional Monte Carlo test of CSR using quadrat counts
Pearson X2 statistic data:  P
X2 = 59.2, p-value = 0.386
alternative hypothesis: regular
Quadrats: 8 by 8 grid of tiles
```

Another test of CSR is the Kolmogorov-Smirnov test in which one compares the observed and expected distributions of the values of certain function $f(x,y)$. First, we specify a real-valued function $f(x,y)$ defined in the window of a point pattern. Then, we calculate this function at each of the data points and compare this empirical distribution of values of f with the predicted distribution of values of f under CSR, using the classical Kolmogorov-Smirnov test. For example, taking $f(x,y) = x$ we get

```
> cdf.test(P, "x", test="ks")
Spatial Kolmogorov-Smirnov test of CSR in two
dimensions data: covariate x evaluated at points
```

```
of P and transformed to uniform distribution
under CSR
D = 0.1725, p-value = 0.1643
alternative hypothesis: two-sided
```

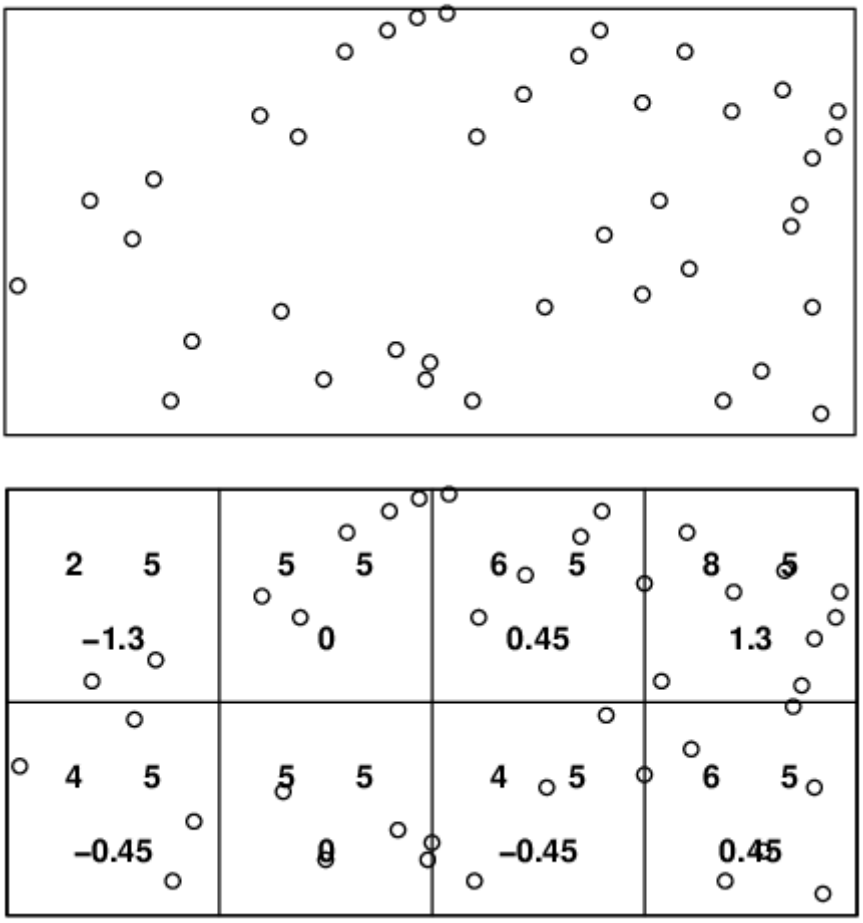

Figure 6. Illustration of a χ^2 test of CSR based on quadrat counts: point pattern (top), quadrat counts (bottom). Plots were prepared in R

If $f(x,y) = y$, we have D $= 0.097018$ and p-value $= 0.811$. This shows, see Fig. 7, that the given point pattern is more uniformly distributed in the y-coordinate direction than in the x-coordinate. Overall, no substantial evidence against H_0 (P is CSR).

Now a few words about the K-function. This is another tool of spatial statistics used for analyzing correlation in point patterns. First observe that a homogeneous Poisson process defined on $\mathbb{R}^d$ is stationary. It means that for each $n = 1,2,\ldots$ and any subsets $A_1,\ldots,A_n$ from $\mathscr{B}(\mathbb{R}^d)$ of finite measure, the joint distributions of vectors

$$(N(A_1),\ldots,N(A_n)) \quad \text{and} \quad (N(A_1+x),\ldots,N(A_n+x))$$

are the same for any $x \in \mathbb{R}^d$. Recall that $N(A) = |\Pi \cap A|$. The Ripley's K-function for a stationary point process Π is defined by

$$K(r) = \frac{1}{\lambda}\mathbb{E}\left[|\Pi \cap B(u,r)\backslash\{u\}| u \in \Pi|\right],$$

where $B(u,r)$ is a ball of radius r centered at u. Note that for a homogeneous Poisson process, the fact that $u \in \Pi$ does not affect the other points of the process, so that $X\backslash\{u\}$ is conditionally a Poisson process. Since the expected number of points in $B(u,r)$ equals $\lambda\pi r^2$ (in $\mathbb{R}^2$), the K-function in this case is

$$K(r) = \pi r^2, \quad r > 0$$

and does not depend on λ. The K-function is nondecreasing for $r > 0$ and converges to 0 as $r \to \infty$, see e.g. Daley and Vere-Jones (2003). The Ripley's function is useful in studying stationary isotropic processes because it then provides a succinct summary of the second-order properties of the process. However the K-function does not completely characterize the point process. For example there exist point processes whose K-functions are equal to πr^2 and the processes are not Poisson processes and so there is dependence between the points, see e.g. Baddeley, Rubak and Turner (2016). Below is the description of a command `Kest(P)` for calculation and plotting of the Ripley's function for a point pattern P:

```
> u <- Kest(P)
> u
Function value object (class fv) for the function r -> K(r)
Entries:
id label description
-- ----- -----------
r r distance argument r
theo Kpois(r)   theoretical Poisson K(r)
border Kbord(r) border-corrected estimate of K(r)
trans Ktrans(r) translation-corrected estimate
of K(r)
iso Kiso(r)     Ripley isotropic correction  estimate of K(r)
--------------------------------------
```

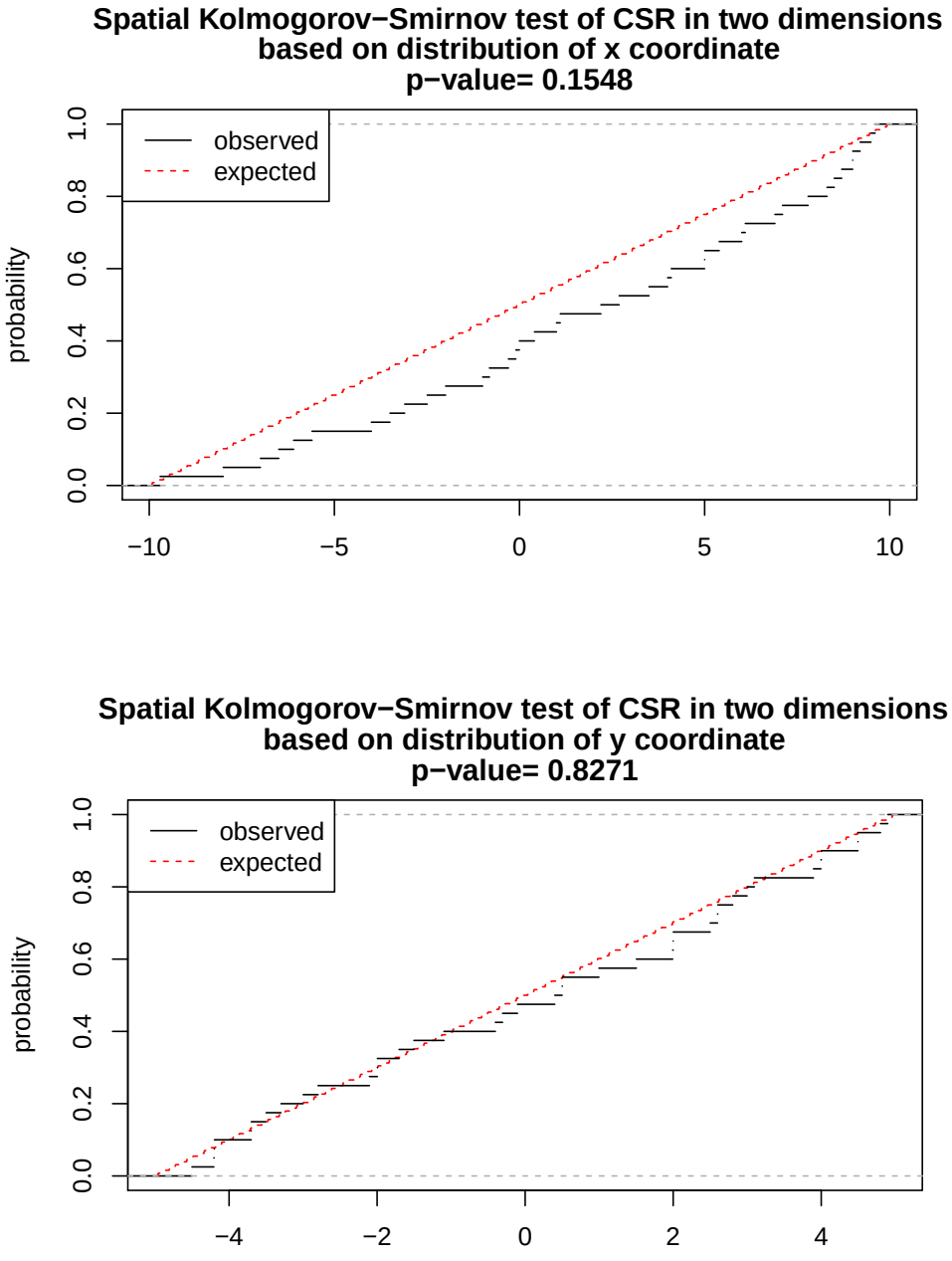

Figure 7. The K-S tests show that considered point pattern is more uniformly distributed in the y-direction than in the x-direction

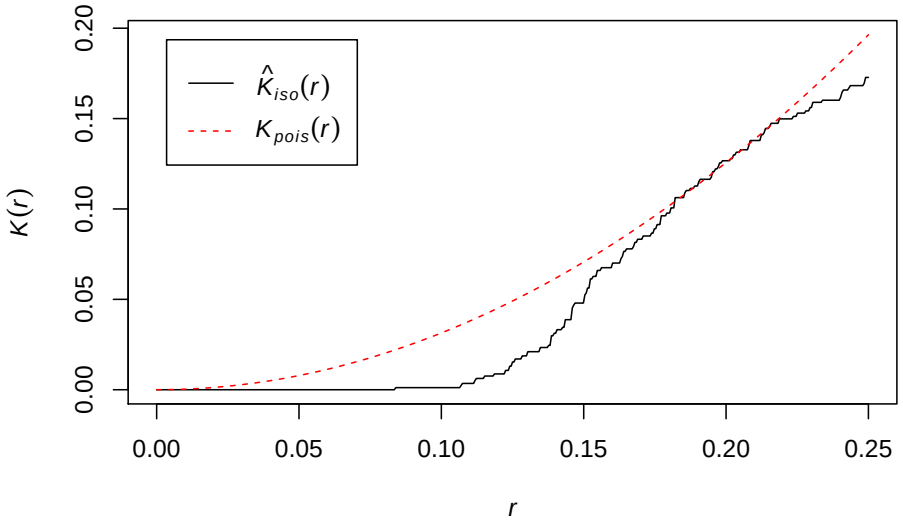

Figure 8. Estimation of the Ripley's K-function in R for the point pattern shown in Fig. 6

6 Cox processes

Before formal definition: if Π is a Poisson point process with intensity function $\lambda(x)$ and we randomize λ, then we get a Cox process, so to speak. Sometimes a Cox process is also called a doubly stochastic Poisson process, but not that often. This is because it has two "sources" of randomness: one is from intensity and the second from a Poisson process. We give the formal definition, see e.g. Grandell (1976) or Kingman (1993) for details. Assume that $(S, \mathscr{B}(S))$ is a state space and let μ be a random nonatomic measure on $\mathscr{B}(S)$. A random countable subset Π of S is said to be a Cox process associated with μ if the conditional distribution of Π given μ is that of Poisson with measure μ. As in (1) it is often assumed that μ is given by a pdf, however due to randomness denoted usually by Λ. Thus

$$\mu(A) = \int_A \Lambda(x)dx \quad \Leftrightarrow \quad \left(\mu(A;\omega) = \int_A \Lambda(x;\omega)dx\right)$$

and we omit ω (also in $N(A;\omega)$, etc). It should be noted that $\Lambda(x)$ is in fact a real-valued random process defined on S. If $A_1,\ldots,A_n$ are pairwise disjoint, then by definition

$$\mathbb{P}\left[N(A_1) = k_1,\ldots,N(A_n) = k_n|\mu\right] = \prod_{i=1}^{n} \frac{\mu^{k_i}(A_i)}{(k_i)!} e^{-\mu(A_i)},$$

where $k_i = 0,1,2\ldots$, for $i = 1,\ldots,n$. From this we obtain

$$\mathbb{P}(N(A_1) = k_1,\ldots,N(A_n) = k_n) = \mathbb{E}\left[\prod_{i=1}^{n} \frac{\mu^{k_i}(A_i)}{(k_i)!} e^{-\mu(A_i)}\right] \tag{7}$$

and, in particular, the distribution of $N(A)$ has the form

$$\mathbb{P}(N(A) = k) = \mathbb{E}\left[\frac{\mu^k(A)}{k!} e^{-\mu(A)}\right], \quad k = 0,1,2,\ldots \tag{8}$$

Incidentally, a Cox process has not the independence property due to (7) and the distribution of $N(A)$ is not that of Poisson by (8). Furthermore, see e.g. Grandell (1976) or by direct computation, we have

$$\begin{cases} \mathbb{E}[N(A)] = \mathbb{E}[\mu(A)] = \int_A \mathbb{E}[\Lambda(x)]dx, \\ \operatorname{var}[N(A)] = \mathbb{E}[N(A)] + \operatorname{var}[\mu(A)], \\ \operatorname{cov}(N(A),N(B)) = \mathbb{E}[\mu(A\cap B)] + \operatorname{cov}(\mu(A),\mu(B)), \end{cases} \tag{9}$$

for bounded $A,B \in \mathscr{B}(\mathbb{R}^d)$. From (9) we have $\operatorname{var}[N(A)] \geq \mathbb{E}[N(A)]$.

We will investigate here, when the distribution of $N(A)$ is close to a discrete uniform. Recall that if a,b are integers and $a < b$ then the probability mass function of a discrete uniform distribution is given by $f(x) = 1/(b-a+1)$, for $x = a, a+1,\ldots,b$. Notation for this is $\mathscr{U}\{a,b\}$. We have

$$\mathbb{E}(X) = \frac{a+b}{2}, \quad \operatorname{var}(X) = \frac{(b-a+1)^2-1}{12}.$$

Example 2 *Given $p \in (0,1)$, consider X with the binomial distribution*

$$\mathbb{P}(X = k|p) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0,1,\ldots,n, \tag{10}$$

denoted by $B(n,p)$. Suppose that p is random with values in $\{p_1,\ldots,p_m\}$ and probabilities

$$\mathbb{P}(p = p_i) = \frac{1}{m}, \quad i = 1,2,\ldots,m.$$

For fixed p the most probable value k_0 of X fulfills the inequality $(n+1)p - 1 \leq k_0 \leq (n+1)p$. Thus if $(n+1)p$ is not an integer then $k_0 = [(n+1)p]$. From (10) we obtain the distribution of X. Namely

$$\mathbb{P}(X = k) = \frac{1}{m}\binom{n}{k} \sum_{i=1}^{m} p_i^k (1-p_i)^{n-k}, \quad k = 0,1,\ldots,n. \tag{11}$$

Hence $\mathbb{E}(X) = \frac{1}{m}\sum_{i=1}^{m} np_i$. In Fig. 9 we present three distributions, each for $n = 10$ and different sets of p_i's. In the first case $p_1 = 0.2$, $p_2 = 0.8$. In the second case $p_1 = 0.2$, $p_2 = 0.5$, $p_3 = 0.8$ and in the third one $p_1 = 0.2$, $p_2 = 0.4$, $p_3 = 0.6$, $p_4 = 0.8$.□

Example 3 *Suppose that $\Lambda = \{\lambda_1, \ldots, \lambda_9\}$ consists of 9 positive numbers. First we choose λ_i, $i = 1, \ldots, 9$, with probability given in the uncaptioned table below. For fixed λ_i we may simulate a realization of a Poisson process on $S = [-30, -10]$ with mean measure $\mu_i(A) = \lambda_i |A|$. That is the way we construct a realization of a Cox process on S.*

λ_i	0.05	0.25	0.5	0.75	1
p_i	0.055	0.118	0.118	0.118	0.118
λ_i	1.25	1.5	1.75	2	
p_i	0.118	0.118	0.118	0.118	

For $A \subset [-30, -10]$, the distribution of $N(A)$ is given by

$$\mathbb{P}(N(A) = k) = \sum_{i=1}^{9} p_i e^{-\lambda_i |A|} \frac{(\lambda_i |A|)^k}{k!}, \quad k = 0, 1, 2, \ldots \tag{12}$$

Hence $\mathbb{E}N(A) = |A| \sum_{i=1}^{9} p_i \lambda_i \approx 1.06 \cdot |A|$. In particular $\mathbb{E}N(S) \approx 21.30$, $\mathbb{E}N^2(S) = 623.44$ and $var[N(S)] \approx 169,75$. Furthermore $\mathbb{P}(N(S) > 40) \approx 0.078$. We put those data in the uncaptioned 3-row table below.

	$N(S)$	$\mathscr{U}\{0,40\}$	$N(B)$	$\mathscr{U}\{0,20\}$
mean	21.30	20	10.65	10
var	169.75	140	47.72	36.66

Example 4 *Suppose that $A \in \mathscr{B}(\mathbb{R}^d)$ and a random measure μ has the form*

$$\mu(A; \omega) = X(\omega)|A|, \quad |A| < \infty,$$

where $X \geq 0$ is a random variable. By (8) we have

$$\begin{aligned} \mathbb{P}(N(A) = k) &= \mathbb{E}\left[\frac{X^k |A|^k}{k!} e^{-X|A|}\right] = \frac{|A|^k}{k!} \mathbb{E}\left[X^k e^{-X|A|}\right] \\ &= \frac{|A|^k}{k!} \int_0^\infty x^k e^{-|A|x} f(x) dx, \qquad k = 0, 1, 2, \ldots \end{aligned}$$

where $f(x)$ is a pdf of X. If $f(x) = e^{-x}$, $x \geq 0$, then for $k \geq 1$ we have

$$\int_0^\infty x^k e^{-(|A|+1)x} dx = \frac{1}{(|A|+1)^{k+1}} \int_0^\infty t^k e^{-t} dt = \frac{k!}{(|A|+1)^{k+1}},$$

by the fact that $\mathbb{E}(X^k) = k!$, $k \geq 1$. Finally

$$\mathbb{P}(N(A) = k) = \frac{1}{|A|+1} \left(\frac{|A|}{|A|+1}\right)^k, \quad k = 0, 1, 2, \ldots$$

meaning that $N(A)$ has geometric distribution.

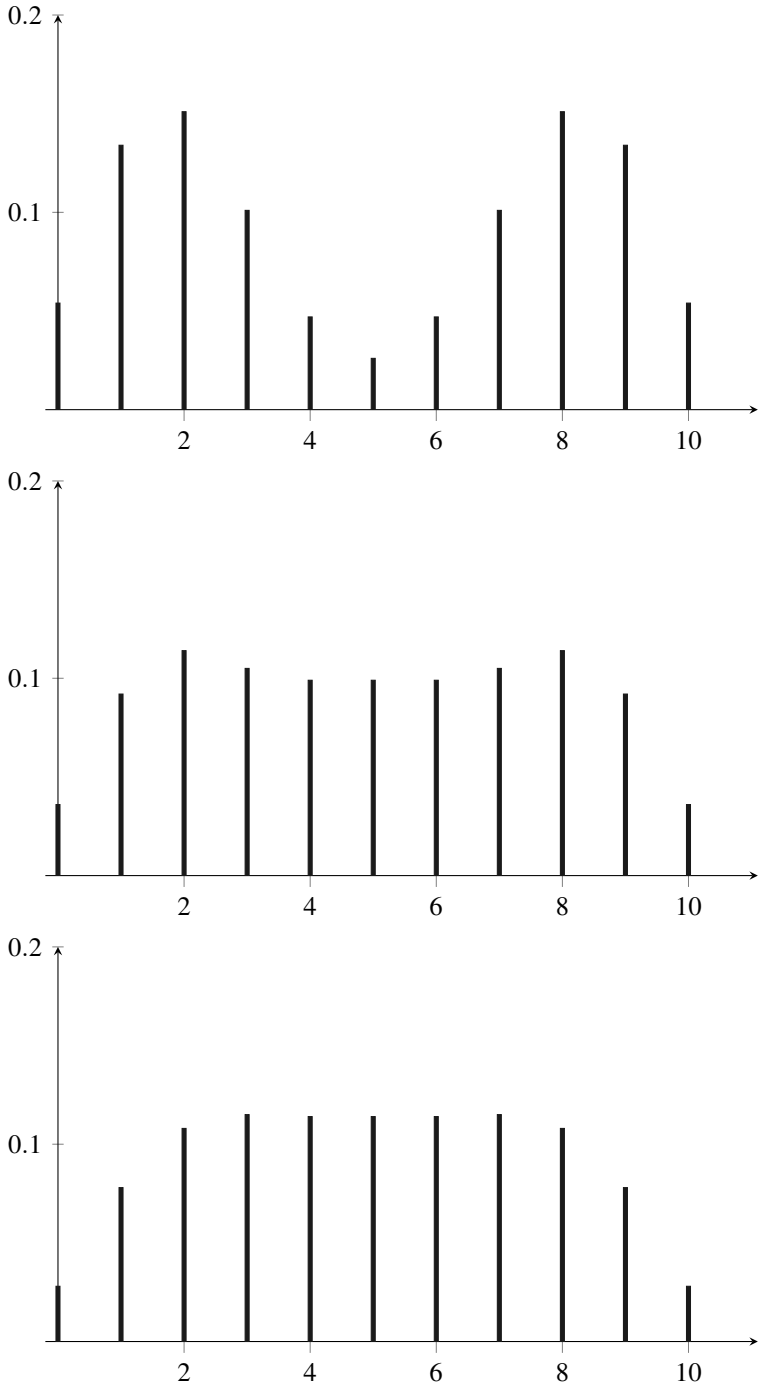

Figure 9. Distributions of X given by (11) for three sets of p_i's. In each case $n = 10$. The last one (right) is "close" to $\mathscr{U}\{0,10\}$

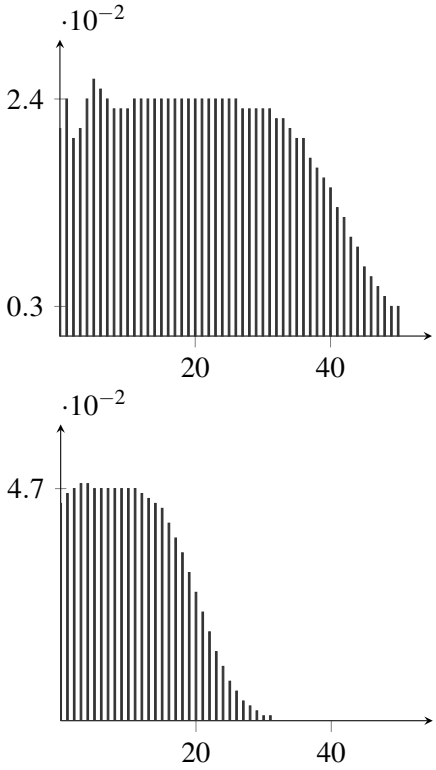

Figure 10. Distributions of $N(A)$ given by (12) for $A = [-30,-10]$ and $[-20,-10]$

7 Appendix: Wasserstein distance and time

Inspired by the book by Julian Barbour, *The End of Time*, see Barbour (1999), we present here the author's own remark about physical time. This opinion is based on intuition derived from dealing with stochastic processes and is stated from a mathematical point of view, i.e. no notions about energy, momentum, mass, etc.

Recall that by a point process we understand (up to technical stuff) a collection of plates with black dots like this one in Fig. 11, where dots represent points in a phase space. This is just a random element in mathematical nomenclature. There is no "time" given there a priori. In a sense, it is a "static process". However when we think of an "ordinary" stochastic process, it is usually a family of random variables (or random elements) X_t indexed by $t \in [a,b]$. And this is the place where t is called "time" because an interval $[a,b]$ is linearly ordered set and we can say that t_1 is before t_2 if $t_1 < t_2$. Hence we have the past $\mathscr{F}_{\leq t}$, the present $\mathscr{F}_{=t}$ and the future $\mathscr{F}_{\geq t}$. On the other hand, this X_t can be treated as a random element in the following sense: $X(\omega) = f_\omega(t)$, where $t \in [a,b]$. In other words: ω is mapped into a function, usually a continuous one, but jumps are also possible, see again Fig. 11.

In this situation t as "time" appears only in a configuration space. Thus "time" is a kind of measure of "change". But what is "change"? Roughly speaking, it is any difference between two configurations in a phase space. Hence, as it is suggested in Barbour (1991), we can define "time" or "duration" for a point process by

```
time, duration := a distance between
                  configurations in a phase space
```

because a phase space has usually "nice" properties, e.g. it may be equipped with a metric. How can we do that? And here is the author's suggestion: we can use the Wasserstein distance.

We formulate this precisely. Suppose that (S,d) is a Polish metric space and $p \in [1,+\infty)$. Let μ and ν be probability measures on S. The Wasserstein distance of order p between μ and ν is defined as follows

$$W_p(\mu,\nu) := \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{S\times S} (d(x,y))^p d\gamma(x,y) \right)^{1/p},$$

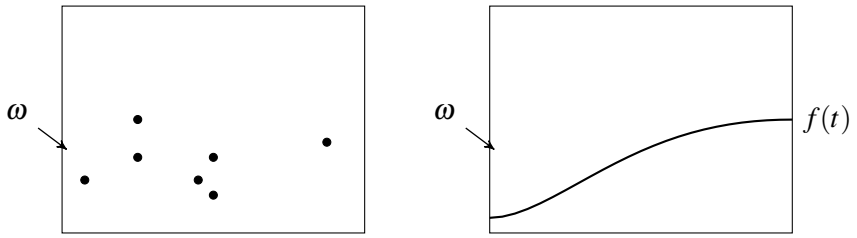

Figure 11. A sample "path" of a point process Π is a countable subset of S. For a stochastic process X, a "path" or a "trajectory" is usually a continuous function

where $\Gamma(\mu,\nu)$ denotes the set of all couplings of μ and ν, i.e. all probability measures on $S\times S$ with marginals μ and ν. For more about the Wasserstein distance and its properties see e.g. Villani (2008). For example, if $S=\mathbb{R}^d$, $p=1$ and $d(x,y)$ is the usual Euclidean distance then

$$W_1(\mu,\nu) = \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{S\times S} ||x-y|| d\gamma(x,y).$$

Now we are ready to define the distance between configurations of a point process. If Π is a random subset of S and $\Pi(\omega)=\{x_1,\ldots,x_k\}$, $\Pi(\omega')=\{x'_1,\ldots,x'_l\}$ we define

$$\mathrm{dist}(\Pi(\omega),\Pi(\omega')) := W_1(\mu,\mu'),$$

where $\mu = (\delta_{x_1}+\ldots+\delta_{x_k})/k$ and $\mu' = (\delta_{x'_1}+\ldots+\delta_{x'_l})/l$. In general, one may take $\mu = \sum_{i=1}^k p_i\delta_{x_i}$ and $\mu' = \sum_{i=1}^l p'_i\delta_{x'_i}$ for discrete probability distributions

$$P=(p_1,\ldots,p_k),\ P'=(p'_1,\ldots,p'_l).$$

Example 5 *Take $S=\mathbb{R}$, configurations $\{0,2\}$, $\{1,5\}$ and*

$$\mu = p\delta_0 + (1-p)\delta_2, \quad \nu = q\delta_1 + (1-q)\delta_5$$

where $p,q \in [0,1]$. The general coupling of μ, ν has the form

$$\begin{aligned}\gamma_a(\mu,\nu) = a\delta_{(0,1)} + (p-a)\delta_{(0,5)} + (q-a)\delta_{(2,1)} \\ + (1-p-q-a)\delta_{(2,5)},\end{aligned}$$

where $a \in [0, \min(p,q)]$. Then

$$\begin{aligned}W_1(\mu,\nu) &= \inf_a \int_{\mathbb{R}^2} |x-y| d\gamma_a(x,y) \\ &= \inf_{a\in[0,\min(p,q)]} (-2a+2p-2q+3) \\ &= -2\min(p,q)+2p-2q+3.\end{aligned}$$

If $p=q=\frac{1}{2}$ then $W_1(\mu,\nu)=2$. For $p=1$, $q=0$ we get $W_1(\mu,\nu)=5$ and so on.

References

Baddeley A., Rubak E., Turner R., (2016), *Spatial Point Patterns. Methodology and Applications with R*, Chapman and Hall/CRC.

Barbour J., (1980), *The End of Time. The Next Revolution in Physics*, Oxford University Press.

Cressie N., (1991), *Statistics for Spatial Data*, Wiley.

Daley D.J., Vere-Jones D., (2003), *An Introduction to the Theory of Point Processes*, Vol. I, Springer.

Diggle P.J., (2014), *Statistical Analysis of Spatial and Spatio-Temporal Point Patterns*, CRC Press.

Grandell J., (1976), *Doubly Stochastic Poisson Processes*, Springer.

Karr A.F., (1991), *Point Processes and their Statistical Inference*, Marcel Dekker.

Kingman J. F. C., (1993), *Poisson Processes*, Oxford.

Long E. J., Hargravc G. K., Cooper J. R., Kitchener B. G. B., Parsons A. J., Hewett C. J. M., Wainwright J., (2014), *Experimental investigation into the impact of a liquid droplet onto a granular bed.*, Physical Review E 89(3-1): 032201.

Ripley B.D., (2014), *Spatial Statistics*, Wiley.

Streit R.L., (2010), *Poisson point processes*, Springer.

Villani C., (2008), *Optimal transport, Old and New*, Springer.

Wentzell A. D., (1980) *Wykłady z teorii procesów stochastycznych*, PWN, Warszawa.

Przemysław Kowalik[1]

A chance-constrained blending problem with amounts of components available in fixed-size portions

Keywords: blending problem, linear programming, uncertainty, stochatic parameters, chance constrained programming, nonlinear programming, mixed integer programming.

Abstract

Linear programming (LP) is a technique used in blending optimization problems since the very beginnings of operational research. However, if not all assumptions necessary to create a valid LP model are satisfied, other techniques must be used including nonlinear programming, stochastic programming or mixed integer linear programming. This paper considers a case of blending optimization in which uncertainty of some parameters occur as well as components of the blend ae available in fixed-size portions (i.e. in discontinuous amounts what results in occurrence of a piecewise-constant cost function). This case can be modelled as a problem of stochastic, more precisely chance-constrained mixed-integer programming problem. It was shown that under an additional assumption of normality of distribution of nondeterministic parameters the problem can be converted to an equivalent mixed-integer nonlinear programming problem or approximated by an (easier to solve) mixed-integer linear programming problem.

1 Introduction

Classical blending optimization models are an important class of linear programming (LP) models. They date back to the very beginnings of the domain of science known as operational or operations research (Stigler (1945), Charnes, Cooper and Mellon (1952)). The most popular models are formulated to determine the cheapest blend of available multi-ingredient components (multi-ingredient raw materials). The blend must satisfy certain requirements on amounts of the ingredients. These requirements may be expressed either in absolute units (mass, volume) or in relative units (usually as percentages). LP blending models can be applied in agriculture to optimal feeding the livestock (a diet problem) and usage of fertilizers as

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well as in production planning in many industries (petroleum, chemical, food processing, metallurgy). Modelling blending optimization problems as LP problems has proved its advantages in numerous cases but also showed its limitations. Creating an accurate mathematical model of a "fragment" of the real world requires satisfying some assumptions related with that model. In case of LP models there are four assumptions:

- additivity and proportionality (together called linearity) between variables describing amounts of the modelled objects and parameters of the model;
- divisibility of variables (they can attain any real value);
- certainty of parameters (they are fixed numbers, not random variables).

Linearity in blending models can be violated by nonlinear relationships between amounts of components and their total price and/or by some physical or chemical processes occurring during the blending.

Lack of divisibility may be caused by lack of availability of some components in any arbitrary amounts but in fixed-size portions only. This fact results in possible extra cost of purchasing the components since the amount of the component needed may not be a exact multiple of the size of the portion of a component.

Certainty of parameters may not occur if unit prices of the components vary randomly and/or amounts of ingredients in the components are not fixed.

The key result of this paper is creating a blending model which takes into account both lack of divisibility caused by availability of components in fixed-size portions only and uncertainty of contents of ingredients in the components included in the blend.

2 A stochastic extension of the deterministic model

A stochastic extension of a deterministic blending model presented below is based on the one proposed in Sakallı and Birgören (2009) and Sakallı, Baykoç and Birgören (2011) for a specific application in metallurgy (brass production). The stochastic model in Sakallı, Baykoç and Birgören (2011) is formulated generally enough, however, to be applicable in any blending optimization. Assumptions and notations are shown below.

There are n types of components (raw materials) and m types of ingredients included in them. Classical blending models assume that percentages of the ingredients are deterministic. However, in some real world applications of blending models the above assumption may not be satisfied. Among components, some of them may have not constant but stochastic (random) percentages of the ingredients.

Let as assume that q components (indexed $1,2,\ldots,q$) have deterministic percentages of the ingredients whereas $n-q$ components (indexed $q+1,q+2,\ldots,m$) have stochastic percentages of the ingredients.

Unless specified otherwise, indexes take the following values

$$i=1,2,\ldots,m, \quad j=1,2,\ldots,n.$$

Let us denote

- c_j – unit cost of the component j (measured in currency units per weight unit e.g. per kilogram);
- $a_{ij},\ j=1,2,\ldots q$ – percentage of the ingredient i in the component j (deterministic);
- $A_{ij},\ j=q+1,q+2,\ldots,j$ – percentage of the ingredient i in the component j (stochastic - a random variable);
- b_i – lower limit on the percentage of the ingredient i in the blend;
- d_i – upper limit on the percentage of the ingredient i in the blend;
- v_j – yield coefficient for the component j $(v_j \in (0,1])$;
- w_i – yield coefficient for the ingredient i $(w_i \in (0,1])$;
- M_j – maximal allowed amount of the component j (optional);
- N_j – minimal allowed amount of the component j (usually 0);
- D – required quantity of blend to be produced.

Decision variables are the following

- x_j – amounts of components to be included in the blend (measured in weight units e.g. kilograms).

The optimization model under consideration is the following

$$\sum_{j=1}^{n} c_j x_j \to \min \tag{1}$$

subject to

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j A_{ij} x_j \leq b_i D \tag{2}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j A_{ij} x_j \geq d_i D \tag{3}$$

$$\sum_{j=1}^{n} x_j = D \tag{4}$$

$$x_j \leq M_j \tag{5}$$

$$x_j \geq N_j \tag{6}$$

where in (2) and (3) $i = 1,2,\ldots,m$ and in (5) and (6) $j = 1,2,\ldots,n$. The objective function (1) denotes the total cost of the components.

Groups of constraints (2) and (3) enforce the lower and upper limits on the percentages of ingredients of the blend, respectively. Some ingredient-related lower or upper limits on the percentages of ingredients may not be specified explicitly. In that case lower limits can be considered as 0 (or 0%) and upper limits as 1 (or 100%) as the content of any ingredient cannot be beyond those bounds. Those constraints are the part of the model which makes it different from any version of a deterministic model since parameters A_{ij} are stochastic.

Constraint (4) ensures producing required quantity of the blend.

Groups of constraints (5) and (6) are used to limit amounts of the components used. Upper limits on amounts of the components used may result from limited availability of components (stock volume). Both lower and upper limits may be caused by technological reasons. If lower limits on amounts of the components are not specified as positive numbers, then they must be set to zero (no negative amounts of components allowed). If upper limits on amounts of the components are not specified, they just can be omitted because there are no "natural upper limits" as zeroes are for lower limits. Yield coefficients (v_j for components, w_i for ingredients) are numbers from the interval $[0,1]$ which represent how much of each component and ingredient is retained during the blending process. Numbers $1 - v_j$ and $1 - w_i$ then obviously represent fractions of components and ingredients lost during the process of blending. Those losses (e.g. partial evaporation) result from technological reasons (in fact, from physical and chemical features of the process of blending). If losses can be neglected, that the yield coefficients are equal to 1 and do not occur in the model explicitly. A detailed discussion on these coefficients can be found in Kim and Lewis (1987).

The model presented above "moves" the mathematical representation of optimization of blending from being a linear programming problem to become a chance-constrained programming problem. Chance-constrained programming was first introduced in Charnes and Cooper (1959) and Miller and Wagner (1965). Its main feature consists in the stochastic form of at least some constraints. More precisely, the probability of complying with constraints for feasibility at the confidence level α is taken into account (see Li, Arellano-Garcia and Wozny (2008)).

A practical way of solving problems formulated as chance-constrained programming model consists in converting those models an into equivalent deterministic ones (see Taha (1997)). This conversion can be performed relatively easy under the assumption of a normal distribution of the stochastic parameters in the constraints. Let us consider the following stochastic constraint

$$P\left\{\sum_{j=1}^{n} a_j x_j \leq b\right\} \geq 1-\alpha. \tag{7}$$

Such a formulation means that the constraint is realized with a minimum probability of $1-\alpha$. Let all a_j be normally distributed and independent stochastic parameters $a_j\ N(\mu_j,\sigma_j^2)$. Let Φ denote the cumulative distribution function of $N(0,1)$ and Φ^{-1} its inverse function. Let $K_{1-\alpha}=\Phi^{-1}(1-\alpha)$. Then (7) can be converted to the following deterministic constraint

$$\Phi\left(\frac{b-\sum_{j=1}^{n}\mu_j x_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2}}\right) \geq \Phi\left(K_{1-\alpha}\right)$$

which is, because of monotonocity of Φ, equivalent to the following inequality

$$\sum_{j=1}^{n}\mu_j x_j + K_{1-\alpha}\cdot\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2} \leq b \tag{8}$$

Obviously, a stochastic constraint with the "$\geq b$" inequality is converted into a deterministic constraint analogical to (8), but with inequality "$\geq b$" and the coefficient $-K_{1-\alpha}$.

When applying (8) to the original stochastic blending problem, it is necessary to assume that all A_{ij} have independent normal distributions $N(\mu_{ij},\sigma_{ij}^2)$. Then the groups of constraints (2) and (3) can be expressed in equivalent, deterministic forms (9) and (10), respectively

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j - K_{1-\alpha}\cdot\sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \geq b_i D \tag{9}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{1-\alpha}\cdot\sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \leq d_i D \tag{10}$$

where $i=1,2,\ldots,m$.

For some ingredients, either lower or upper limit on percentage of a ingredient may not exist. More precisely, the limits are not expressed explicitly, but is was stated before, no lower limit specified stands implicitly for $b_i=0$ (or 0%) and no upper limit specified stands implicitly for $d_i=1$ (or 100%). In case when both lower and upper limits are stated explicitly (and obviously neither of them is the

extreme one, i.e. $b_i > 0$ and $d_i < 1$), some corrections must be made. If both lower and upper limits on percentage of a ingredient exist, the chances of attaining by the random variables values greater or lower than the mean parameter are equal. Therefore $1-\alpha$ must be assigned to both the upper and lower limits with equal probabilities. Below a formulation of the problem in a more compact form than that originally created in Sakallı, Baykoç and Birgören (2011) is presented.

Let $K_{1-\alpha/2} = \Phi^{-1}(1-\alpha/2)$. Then the corrected versions of (9) and (10), respectively, are the following

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j - K_{1-\alpha/2} \cdot \sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \geq b_i D \tag{11}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{1-\alpha/2} \cdot \sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \leq d_i D \tag{12}$$

for such i for which $b_i > 0$ and $d_i < 1$.

More precisely, if one limit on percentage (either lower or upper) is defined explicitly for a given ingredient i, then the corresponding constraint must be used the standard deviation term multiplied by the coefficient equal to either $-K_{1-\alpha}$ or $K_{1-\alpha}$, respectively. If both specification limits are given, however, then the coefficients $-K_{1-\alpha/2}$ and $K_{1-\alpha/2}$ are used instead. In order to formulate the entire model in the compact form, the exact values of the coefficients which are multiplied by the standard deviation terms will be defined conditionally for all combinations of b_i and d_i. The following notations will be used

- K_{α,b_i} – coefficient to be used in the constraint for the lower limit of ingredient i;
- K_{α,d_i} – coefficient to be used in the constraint for the upper limit of ingredient i.

"Conditional definitions" of K_{α,b_i} and K_{α,d_i} are the following

$$K_{\alpha,b_i} = \begin{cases} 0 & b_i = 0,\ d_i < 1 \\ -K_{1-\alpha/2} & b_i > 0,\ d_i < 1 \\ -K_{1-\alpha} & b_i > 0,\ d_i = 1 \end{cases} \tag{13}$$

$$K_{\alpha,d_i} = \begin{cases} K_{1-\alpha} & b_i = 0,\ d_i < 1 \\ K_{1-\alpha/2} & b_i > 0,\ d_i < 1 \\ 0 & b_i > 0,\ d_i = 1 \end{cases}$$

where $i = 1, 2, \ldots, m$.

Obviously, the combination $b_i = 0$ and $d_i = 1$ is not considered because the content of each ingredient cannot take the value beyond the $[0,1]$ interval. In this

case, a pair of constraints with such lower and upper limits on ingredients, respectively, is satisfied by all x_j satisfying (4), (5) and (6) so the constraints do not affect the feasible set, and, obviously, an optimal solution.

Finally, the stochastic model defined in (1) to (6) at the confidence level α and the additional assumption that all A_{ij} have independent normal distributions $N(\mu_{ij}, \sigma_{ij}^2)$ can be expressed as the following nonlinear programming model

$$\sum_{j=1}^{n} c_j x_j \to \min \tag{14}$$

subject to

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{\alpha,b_i} \cdot \sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \geq b_i D \tag{15}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{\alpha,d_i} \cdot \sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \leq d_i D \tag{16}$$

$$\sum_{j=1}^{n} x_j = D \tag{17}$$

$$x_j \leq M_j \tag{18}$$

$$x_j \geq N_j \tag{19}$$

where in (15) and (16) $i = 1, 2, \ldots, m$ and in (18) and (19) $j = 1, 2, \ldots, n$.

The meaning of the objective function (14) as well as of the constraints (17), (18) and (19) is identical to the meaning of (1), (4), (5) and (6), respectively. Constraints (15) and (16) are the replacements for (2) and (3) with either (9) and (10) or (11) and (12) (depending on values of b_i and d_i), respectively. They reflect uncertainty of contents of ingredients in some components.

Nonlinear terms in (15) and (16) make the problem hard to solve. Since the linear programming optimization software is widely available and, what is even more important, it guarantees obtaining (within built-in computational accuracy) an optimal solution, it would be worth considering to perform some kind of linearization of the model. Obviously, since a linearized formulation is not an equivalent but just an approximate one, it can result in a solution which is possibly not optimal (in the sense of the initial problem).

How linearization can be done? The first, "rough" approach consists just in ignoring nonlinear terms and treating expected values of percentages μ_{ij} as deterministic values (simply disregarding uncertainty). A more sophisticated approach does not neglect the risk of deviation of the real percentages of ingredients from

average values. It consists in replacing nonlinear variance-related terms with linear ones. Linear terms used to estimation are chosen in such way that the feasible set of the linearized problem is a subset of the primary problem.

The linear approximation approach presented below is based on Agpak and Gokcen (2007). As the main idea it uses the following inequality

$$\sqrt{\sum_{j=1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \leq \sum_{j=1}^{n} w_i v_j \sigma_{ij} x_j. \tag{20}$$

which is true if variables x_j are all nonnegative (what is the case of blending models) Replacing the nonlinear parameter with a bigger value than its actual value reduces the feasible solution space. Therefore we may obtain an approximate solution rather than optimal.

Again, as in case of the conversion of the initial stochastic model to the deterministic nonlinear one it is necessary to distinguish cases: the lower limit on percentage of an ingredient is not defined explicitly (it is 0%), the upper limit on percentage of an ingredient is not defined explicitly (it is 100%), or both limits are defined explicitly (they are both between 0% and 100%). Previously defined coefficient values K_{α,b_i} and K_{α,d_i} will also be used in linearization of the nonlinear model.

The linearized model is very similar to the model described in (14) to (19). The only difference is that groups of constraints (15) and (16) are replaced, by adapting inequality (20), with linear ones (21) and (22).

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{\alpha,b_i} \cdot \sum_{j=q+1}^{n} w_i v_j \sigma_{ij} x_j \geq b_i \tag{21}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{\alpha,d_i} \cdot \sum_{j=q+1}^{n} w_i v_j \sigma_{ij} x_j \leq d_i \tag{22}$$

Linear constraints (21) and (22) can be expressed in simpler notation

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j (\mu_{ij} + K_{\alpha,b_i} \sigma_{ij}) x_j \tag{23}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j (\mu_{ij} + K_{\alpha,d_i} \sigma_{ij}) x_j \tag{24}$$

The entire linearized model is now formulated as

$$\sum_{j=1}^{n} c_j x_j \rightarrow \min \tag{25}$$

subject to

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j (\mu_{ij} + K_{\alpha,b_i}\sigma_{ij}) x_j \tag{26}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j (\mu_{ij} + K_{\alpha,d_i}\sigma_{ij}) x_j \tag{27}$$

$$\sum_{j=1}^{n} x_j = D \tag{28}$$

$$x_j \leq M_j \tag{29}$$

$$x_j \geq N_j \tag{30}$$

where in (26) and (27) $i = 1,2,\ldots,m$ and in (29) and (30) $j = 1,2,\ldots,n$.

The objective function (14) as well as of the constraints (28), (29) and (30) are identical to (1), (4), (5) and (6) or (14), (17), (18) and (19), respectively. Constraints (26) and (27) are the replacement for (15) and (16), respectively. They reflect uncertainty of contents of ingredients in some components.

The above model is a linear programming problem which can be solved by using widely available LP optimization software.

3 A deterministic blending model with fixed-size portions of components

The assumption of linearity in blending models means from a practical point of view that costs of purchasing and – not explicitly mentioned processing the components are strictly proportional to amounts of the components used for blending. Whereas such an assumption at a first glance seems to be legitimate, it does not need to. First reason for the lack of proportionality between amounts of the components and their costs is connected with pricing policy of suppliers of components. Purchasing large amounts of components may result in some discounts in unit prices. However, unit prices of components may also increase e.g. in case when the capacities of some suppliers are attained and alternative suppliers offer components at higher unit prices. Pricing policy is not the only reason of nonlinearity of the total cost of a blend. Other reasons for nonlinearity can be connected with start-up costs (resulting from minimal cost of purchase or minimal usage of a component) and availability of components in fixed-size portions only (see Chachuat (2009), Rardin(1998)).

Availability of components in portions can be modelled in two variants. The first one assumes full usage of portions of components. Its formulation is relatively simple but it restricts applications to those in which target amount of the

blend is not defined (mainly diet optimization problems) (Lee (2016), Patil and Kasturi (2016)). Another variant "allows" for using any fraction of the portion of a component. In this case it is possible to create models in which target amount of the blend is explicitly defined. Probably, on the the earliest blending model dealing with availability of components in portions is Westerberg, Bjorklund and Hultman (1977). However it is restricted to the case when one portion only can be used.

Some extensions of blending models to use many portions of components can be found in Kowalik (2013, 2018a, 2018b) and Kowalik (2018a, 2018b) will be used as a starting point for further considerations.

There are n types of components (raw materials) and m types of ingredients included in them. Each component is available (in sale) in fixed-size portions only but it can be used in any amounts. In order to simplify the notation reasons, the model considers components available in fixed-size portions only whereas obviously blends can be produced of components available both in fixed-size portions and in arbitrary amounts. If the same component is available in portions of various sizes then each size of a portion defines a new "virtual" component. It is because portions of the same component but of different sizes have the same chemical composition but different unit prices.
Indexes take the following values

$$i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n.$$

Let us denote

- c_j – unit cost of the component j (measured in currency units per weight unit e.g. per kilogram);
- a_{ij} – percentage of the ingredient i in the component j;
- b_i – lower limit on the percentage of the ingredient i in the blend;
- d_i – upper limit on the percentage of the ingredient i in the blend;
- v_j – yield coefficient for the component j ($v_j \in (0,1]$);
- w_i – yield coefficient for the ingredient i ($w_j \in (0,1]$);
- W_j – size of a portion of the component j (measured in weight units e.g. kilograms; can be considered just as net weight of the portion);
- M_j – maximal allowed amount of the component j (optional);
- N_j – minimal allowed amount of the component j;
- G_j – maximal number of available portions of components – stock size (optional);
- D – required quantity of the blend to be produced.

Decision variables:

- x_j – amounts of components to be used in the blend (measured in weight units e.g. kilograms);
- y_j – numbers of portions of components to be used in the blend (measured in weight units e.g. kilograms).

The optimization model under consideration is the following

$$\sum_{j=1}^{n} c_j y_j \to \min \tag{31}$$

subject to

$$\sum_{j=1}^{n} w_i v_j a_{ij} x_j \leq b_i D \tag{32}$$

$$\sum_{j=1}^{n} w_i v_j a_{ij} x_j \geq d_i D \tag{33}$$

$$\sum_{j=1}^{n} x_j = D \tag{34}$$

$$x_j \leq W_j y_j \tag{35}$$

$$x_j \leq M_j \tag{36}$$

$$x_j \geq N_j \tag{37}$$

$$y_j \leq G_j \tag{38}$$

$$y_j - \text{integer} \tag{39}$$

where in (32) and (33) $i = 1,2,\ldots,m$ and in (35), (36), (37) and (38) $j = 1,2,\ldots,n$.

Classical blending models based on linear programming are based on (usually) implicit, yet rather obvious assumption "amounts of components purchased = amounts of components used". This means that the same variables x_j stand for "amounts of components purchased" in the objective function and for "amounts of components used" in the ingredient-related constraints and in the constraint forcing the required amount of the blend. Even if losses of some components or ingredients occur in volumes proportional to their amounts, yield coefficients can reflect this fact without losing the linearity of the model. The key difference between and models in which components are available in fixed-size portions is invalidity of the abovementioned assumption. Amounts of used components grow proportionally to D – required quantity of the blend but its total cost does not since it is necessary to buy components in portions of fixed size and fixed price per portion. That is why

it is necessary to introduce a set of new integer variables y_j which denote numbers of portions of components purchased. Meanwhile variables x_i are interpreted as "amounts of components used" only. The dependence between the amount of each component used and the cost of that component purchased in portion is then a piecewise constant function. An example of the chart of such a function is shown in Fig.1.

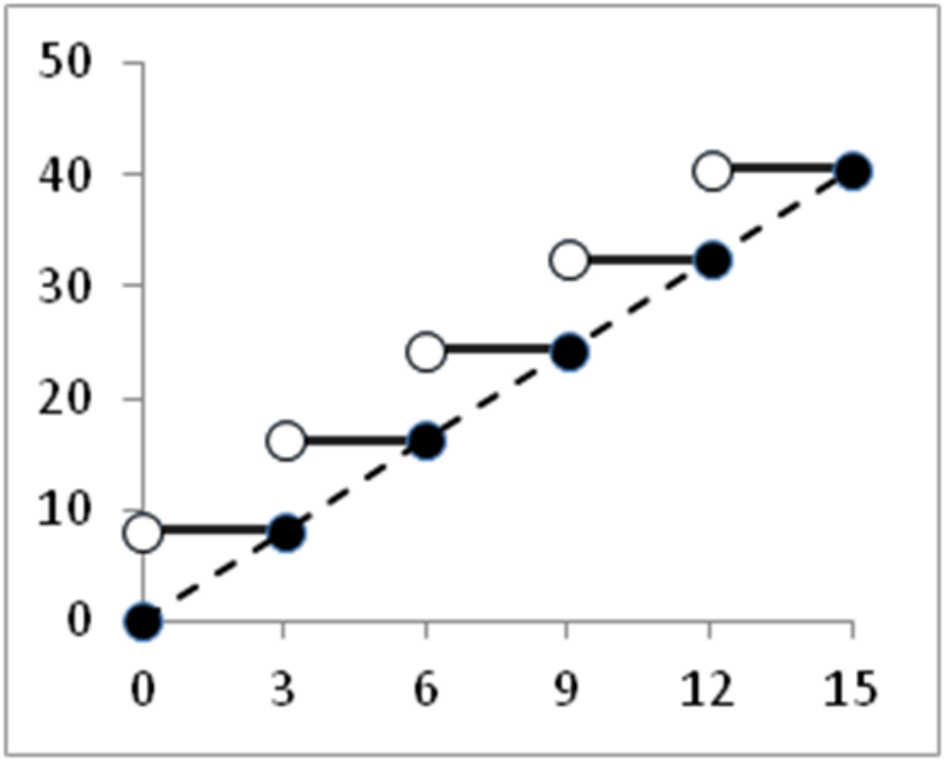

Figure 1. An example of the chart of a piecewise constant function

The objective function (31) denotes the total cost of the components expressed as the total cost of portions in which they are available. Groups of constraints (32) (33), (34), (36) and (37) are analogical to (2), (3), (4), (5) and (6), respectively. Inequalities (35) ensure that the amounts of components used are limited by multiples of the sizes of their portions i.e. by amounts which must be purchased and paid for. Inequalities (38) forces number of purchased portions of components will not excee that the numbers of available portions of components (stock size). Upper limits on amounts of the components (36) now can result from technological reasons only. Limited availability of components (stock size) is represented by (38).

4 A stochastic (chance-constrained) blending model with fixed-size portions of components

The model considered in this chapter is a "fusion" of the ones mentioned in two previous chapters i.e. it takes into account both uncertainty of content of ingredients and availability of components in fixed-size portions. Components available in arbitrary amounts will not be considered for simplicity of the formulation.

There are n types of components (raw materials) and m types of ingredients included in them. Each component is available (in sale) in fixed-size portions only but it can be used in any amounts. Among components, q of them (indexed $1,2,\ldots,q$) have deterministic percentages of the ingredients and $n-q$ components (indexed $q+1,q+2,\ldots,n$) have stochastic percentages of the ingredients. Unless specified otherwise, indexes take the following values

$$i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n.$$

Let us denote

- c_j – unit cost of the component j (measured in currency units per weight unit e.g. per kilogram);
- $a_{ij},\ j = 1,2,\ldots q$ – percentage of the ingredient i in the component j (deterministic);
- $A_{ij},\ j = q+1,q+2,\ldots,n$ – percentage of the ingredient i in the component j (stochastic - a random variable);
- b_i – lower limit on the percentage of the ingredient i in the blend;
- d_i – upper limit on the percentage of the ingredient i in the blend;
- v_j – yield coefficient for the component j ($v_j \in (0,1]$);
- w_i – yield coefficient for the ingredient i ($w_j \in (0,1]$);
- W_j – size of a portion of the component j (measured in weight units e.g. kilograms);
- M_j – maximal allowed amount of the component j (optional);
- N_j – minimal allowed amount of the component j;
- G_j – number of available portions of the component j — stock size (optional);
- D – required quantity of blend to be produced.

Decision variables:

- x_j – amounts of components to be used in the blend (measured in weight units e.g. kilograms);
- y_j – numbers of portions of components to be used in the blend (measured in weight units e.g. kilograms).

Because of assumption of normality and independence of A_{ij}, the optimization model will be formulated at once in the following deterministic form (a mixed integer nonlinear programming problem).

$$\sum_{j=1}^{n} c_j y_j \to \min \tag{40}$$

subject to

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{\alpha,b_i} \cdot \sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \geq b_i D \tag{41}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j \mu_{ij} x_j + K_{\alpha,d_i} \cdot \sqrt{\sum_{j=q+1}^{n} w_i^2 v_j^2 \sigma_{ij}^2 x_j^2} \leq d_i D \tag{42}$$

$$\sum_{j=1}^{n} x_j = D \tag{43}$$

$$x_j \leq W_j y_j \tag{44}$$

$$x_j \leq M_j \tag{45}$$

$$x_j \geq N_j \tag{46}$$

$$y_j \leq G_j \tag{47}$$

$$y_j - \text{integer} \tag{48}$$

where in (41) and (42) $i = 1,2,\ldots,m$ and in (44), (45), (46), (47) and (48) $j = 1,2,\ldots,n$.

The objective function (40) is the same as (31). Groups of constraints (41), (42), (43), (45) and (46) are the same as (15),(16), (17), (18) and (19), respectively (they are adopted from the nonlinear model converted from the stochastic model with arbitrary amounts of components available). Groups of constraints (44), (47) and (48) are the same as (35), (38) and (39), respectively (they are adopted from the deterministic model with components available in portions).

The linearization is the following one

$$\sum_{j=1}^{n} c_j y_j \to \min \tag{49}$$

subject to

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j (\mu_{ij} + K_{\alpha,b_i}\sigma_{ij}) x_j \geq b_i D \tag{50}$$

$$\sum_{j=1}^{q} w_i v_j a_{ij} x_j + \sum_{j=q+1}^{n} w_i v_j (\mu_{ij} + K_{\alpha,d_i}\sigma_{ij}) x_j \leq d_i D \tag{51}$$

$$\sum_{j=1}^{n} x_j = D \tag{52}$$

$$x_i \leq W_j y_j \tag{53}$$

$$x_j \leq M_j \tag{54}$$

$$x_j \geq N_j \tag{55}$$

$$y_j \leq G_j, \tag{56}$$

$$y_j - \text{integer} \tag{57}$$

where in (50) and (51) $i = 1,2,\ldots,m$ and in (53), (54), (55), (56) and (57) $j = 1,2,\ldots,n$.

The objective function (49) is the same as (31) or (40). Groups of constraints (50) and (51) are the same as (26) and (27), respectively (they are adopted from the linearization of the nonlinear model converted from the stochastic model with arbitrary amounts of components available). Groups of constraints (53), (55) and (56) are the same as (43), (45) and (46) or (17), (18) and (19), respectively (they are adopted from the nonlinear model converted from the stochastic model with arbitrary amounts of components available). Groups of constraints (54), (57) and (58) are the same as (35), (38) and (39) or (44), (47) and (48), respectively (they are adopted from the deterministic model with components available in portions). The entire model is a mixed integer linear programming problem.

5 Conclusions

The results presented in this paper show that simple linear blending optimization models which were created in the very beginnings of operational research in the 1940's and 1950's can still be developed. Developments consist in better reflecting real-world conditions of making decisions. Introducing less restrictive assumptions in a model does not usually happen at no cost, however. This "cost" is mainly

connected with the fact that problems created on the basis of more elaborate models require more calculations necessary to solve them. An increasing demand for computational power may be "quantitative" (e.g. replacing simple LP models with more complicated ones) or "qualitative" (i.e. "transition" to another, "more difficult" type of optimization: from linear to nonlinear, from real-number to integer, from deterministic to stochastic).

The blending optimization model considered in this paper simultaneously takes into account two factors: no certainty of content of ingredients in some components and availability of components in fixed-size portions. Under additional assumption of normality and independence of distributions of contents of ingredients, the model can be expressed a stochastic, more precisely a chance-constrained linear stochastic programming problem. The primary model can be converted into an equivalent integer nonlinear programming problem or in an approximate integer nonlinear programming problem. Both of the converted linear problems (especially the latter one) are tractable with existing optimization software

Further investigations on the subject may be performed towards connecting uncertainty of contents of ingredients with other cases of nonlinearity of costs. This includes costs with multiple fixed charges (components are available in arbitrary amounts but there are also some extra fixed costs connected e.g. with engaging trucks for transportation) or fixed costs of preparing a part of the portion of a component (e.g. cutting a fragment of the ingot of some alloy in metallurgy). The exact direction of those investigations should depend mainly on specific needs of practitioners employed in various industries in which blending optimization is necessary.

References

Agpak K., Gokcen H., (2007), *A chance-constrained approach to stochastic line balancing problem*, European Journal of Operational Research, 180, 1098–1115.

Chachuat B., (2009), *Mixed-Integer Linear Programming (MILP): Model Formulation, ChE 4G03: Optimization in Chemical Engineering*, McMaster University, Department of Chemical Engineering, http://macc.mcmaster.ca/maccfiles/ chachuatnotes/ 07-MILP-I_handout.pdf, (access 10.09.2018).

Charnes A., Cooper W. W., Mellon B., (1952), *Blending Aviation Gasolines – A Study in Programming Interdependent Activities in an Integrated Oil Company*, Econometrica, vol. 20, no. 2, 135–159.

Charnes A., Cooper W. W.,n (1959), *Chance-constrained programming*, Management Science, 6, 73–79.

Kim J., Lewis R. L., (1987), *A large scale linear programming application to least cost charging for foundry melting operations*, Transactions of the American Foundrymen's Society, 95, 735–744.

Kowalik P., (2013), *Minimalizacja kosztu mieszanek komponentów dostępnych w opakowaniach o ustalonych pojemnościach*, Rola informatyki w naukach ekonomicznych i społecznych, Innowacje i implikacje interdyscyplinarne, 2/2013, 251–259.

Kowalik P., (2018a), *Minimalizacja kosztu mieszanek jako zadanie programowania liniowego całkowitoliczbowego*, [in:] Ludzie nauki w kręgu interdyscyplinarnych badań, (ed. by Kruzel R., Balina R., Tańska H., Ejdys S., Drewniak M.), Wydawnictwo Naukowe INTELLECT, Waleńczów, 21–32.

Kowalik P., (2018b), *Optymalizacja składu mieszanek ze względnymi normami zawartości składników oraz nieciągłymi przedziałami liniowymi funkcjami kosztów,* [in:] Wyzwania współczesnej nauki cz. 3 (ed. by Ejdys S., Balina R., Szul T., Drewniak M., Balina M.), Wydawnictwo Naukowe INTELLECT, Waleńczów, 94–107.

Lee H.-H., (2016), *Programming with MATLAB 2016*, SDC Publications.

Li P., Arellano-Garcia H., Wozny G., (2008), *Chance constrained programming approach to process optimization under uncertainty*, Computers and Chemical Engineering, 32, 25–45.

Miller L. B., Wagner H., (1965), *Chance-constrained programming with joint constraints*, Operations Research, 13, 930–945.

Patil A. N., Kasturi S., (2016), *Optimal Diet Decision Using Linear Programming*, International Research Journal of Engineering and Technology (IRJET), Volume: 03 Issue: 08, 2197–2199.

Rardin R. L., (1998), *Optimization in Operations Research*, Prentice-Hall, New York.

Sakallı U. S., Birgören B., (2009), *A spreadsheet-based decision support tool for blending problems in brass casting industry*, Computers & Industrial Engineering, 56, 724–735.

Sakallı U. S., Baykoç Ö. F., Birgören B., (2011), *Stochastic optimization for blending problem in brass casting industry*, Annals of Operations Research, 186, 141–157.

Stigler G. J., (1945), *The cost of subsistence*, Journal of Farm Economics, vol. 27, no. 2 (May), 303–314.

Taha H. A., (1997), *Operations Research – an Introduction*, Prentice Hall, New York, 1997.

Westerberg C., Bjorklund B., Hultman E., (1977), *An Application of Mixed Integer Linear Programming in a Steel Mill*, Interfaces 7:2, 39–43.

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Lower and upper partial moments of deformed modified power series distributions

Keywords: deformed distribution, lower partial (incomplete) moments, upper partial (incomplete) moments, factorial partial (incomplete) moments, recurrence relations, modified power series distributions, generalized Poisson distribution, generalized negative binomial distribution, generalized logarithmic series, lost game distributions

Abstract

Recurrence formulae for lower and upper partial moments and lower and upper partial descending and ascending factorial moments of deformed modified power series distributions are derived. The obtained formulae are demonstrated on the deformed generalized binomial, the deformed generalized Poisson, the deformed generalized logarithmic, the deformed lost games distribution and the deformed distribution of the number of customers served in a busy period.

1 Introduction

In observing a random variable, it sometimes happens that some values are incorrectly observed. For example, in determining the number of defects per unit or item examined, an inspector may err by reporting units which actually contain a single defect as being perfect or free of defects. The illustrative application of this situation was studied by Cohen (1960). He considered altered data from Bortkiewicz's (1898) classical example on deaths from the kick of a horse in the Prussian Army. In his example twenty of 200 given records which should have shown one death were in error by reporting no deaths. The same example was considered by Williford and Bingham (1979). They considered classical Poisson distribution assuming some values of one are erroneously observed and reported as being zeros. Moreover, the Poisson distribution belongs to the class of modified power series distributions (MPSD) introduced by Gupta (1974). He obtained the

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recurrence relation between the central moments, the factorial moments of MPSD. Tripathi et al. (1986) obtained the incomplete moments of MPSD. They also obtained the recurrence relation between the incomplete moments about origin and the factorial incomplete moments of MPSD. Hassan and Ahmad (2009) studied MPSD where some of the observations corresponding two are misclassified as one with a positive probability. They obtain recurrence relations among ordinary, central and factorial moments of MPSD deformed at the point two. The formulae and recurrence relations for moments of MPSD deformed at any of the support point were establish by Murat and Szynal (2003).

On the other hand the importance of partial or incomplete moments are well known. In statistical decision theory, computations often involve the partial moments of a random variable. For instance, in finite-action problem with linear loss functions, the expected value of perfect information and the expected value of sample information can be expressed in terms of linear loss integrals, which are simply partial expectation. In Bayesian point estimation problem with linear loss functions, partial expectations are use to determine the optimal decision, or point estimate. Some interesting examples, such as insurance purchasing, Bayesian point estimation, inventory theory, theory of the firm, stopping rules, screening and classifications, are discussed by Winkler et al. (1972). Antle (2010) proposed the use of partial-moment functions as a flexible way to characterize, estimate, and test asymmetric effects of inputs on output as deviations from a reference value, such as the mean or a behaviourally determined threshold. This research contributed to the growing body of literature on methods to characterize agricultural output as a random variable determined by complex interactions between management decisions and exogenous random events such as weather and pests.

The present contribution deals with formulae and recurrence relations for lower and upper partial moments, lower descending and ascending factorial partial moments and upper descending and ascending factorial partial moments of deformed modified power series distribution with special reference to the deformed generalized binomial, the deformed generalized Poisson, the deformed generalized logarithmic, the deformed lost games distribution and the deformed distribution of the number of customers served in a busy period. These moments have been establish with use of the method presented in Gupta (1974) and expressed in terms of its first derivative. From these relations we can get other characteristics of deformed distributions mentioned above, such as the coefficients of skewness and kurtosis which play an important part in describing statistical properties of random variables.

2 Definitions and notations

In this study we are concerned with the case in which a value $s+1$ of MPSD is reported as s, where $s = 0, 1, 2, \ldots$. Suppose the number of defectives actually

present in a sample is a random variable with MPSD. Let α designate the probability that a sample which actually contains $s+1$ defectives is misclassified by reporting it as containing only s defectives. In all other cases we assume correct observation and reporting of defectives. Let X designate the number of defectives reported in a sample. The distribution of random variable X is called **deformed modified power series distribution** (DMPSD) and its probability function is written as

$$\mathbf{P}[X=x]=\begin{cases}\left[1+\alpha g(\theta)\frac{a(s+1)}{a(s)}\right]\frac{a(s)[g(\theta)]^s}{f(\theta)}, & x=s,\\ (1-\alpha)\frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}, & x=s+1,\\ \frac{a(x)[g(\theta)]^x}{f(\theta)}, & x\neq s,s+1,\ x\in N\cup\{0\};\\ & 0<\alpha\leq 1,\end{cases} \tag{1}$$

where $f(\theta)=\sum a(x)[g(\theta)]^x$, $g(\theta)$ is positive, finite and differentiable and coefficients $a(x)$ are free of θ.

The aim of this article is to establish the recurrence relation for moments of X defined in the following definitions.

Definition 1 The *r*-th lower partial moment about a point *c* *(r-th incomplete moment on the left about a point c) is defined by*

$$\mu_r(t,c)=\sum_{x=0}^{t}(x-c)^r\mathbf{P}[X=x]. \tag{2}$$

Definition 2 The *r*-th upper partial moment about a point *c* *(r-th incomplete moment on the right about a point c) is defined by*

$$\mu^r(t,c)=\sum_{x=t}^{\infty}(x-c)^r\mathbf{P}[X=x]. \tag{3}$$

Definition 3 The *r*-th lower partial descending factorial moment about a point *c* *(r-th incomplete descending factorial moment on the left about a point c) is defined by*

$$\mu_{(r)}(t,c)=\sum_{x=0}^{t}(x-c)^{(r)}\mathbf{P}[X=x], \tag{4}$$

where $y^{(r)}=y(y-1)(y-2)\cdots(y-r+1)$.

Definition 4 The *r*-th upper partial descending factorial moment about a point *c* *(r-th incomplete descending factorial moment on the right about a point c) is defined by*

$$\mu^{(r)}(t,c)=\sum_{x=t}^{\infty}(x-c)^{(r)}\mathbf{P}[X=x]. \tag{5}$$

Definition 5 **The *r*-th lower partial ascending factorial moment about a point *c*** *(r-th incomplete ascending factorial moment on the left about a point c) is defined by*

$$\mu_{[r]}(t,c) = \sum_{x=0}^{t} (x-c)^{[r]} \mathbf{P}[X=x], \tag{6}$$

where $y^{[r]} = y(y+1)(y+2)\cdots(y+r-1)$.

Definition 6 **The *r*-th upper partial ascending factorial moment about a point *c*** *(r-th incomplete ascending factorial moment on the right about a point c) is defined by*

$$\mu^{[r]}(t,c) = \sum_{x=t}^{\infty} (x-c)^{[r]} \mathbf{P}[X=x]. \tag{7}$$

Moreover, we will use the following notation for

- ordinary lower and upper partial moments: $\mu_r(t,0) = m_r(t)$, $\mu^r(t,0) = m^r(t)$,
- central lower and upper partial moments: $\mu_r(t,m_1(t)) = \nu_r(t)$, $\mu^r(t,m_1(t)) = \nu^r(t)$,
- factorial descending lower and upper partial moments: $\mu_{(r)}(t,0) = m_{(r)}(t)$, $\mu^{(r)}(t,0) = m^{(r)}(t)$,
- factorial ascending lower and upper partial moments: $\mu_{[r]}(t,0) = m_{[r]}(t)$, $\mu^{[r]}(t,0) = m^{[r]}(t)$.

3 Recurrence relations for lower partial moments of DMPSD

In this Section we consider only the situation when a point of deformation s is less than a limit t because if $t < s$ then the lower partial moments are independent of a point s. When $t \leq s$ then relations for lower partial moments of DMPSD coincide with formulae obtained by Gupta (1974).

Theorem 1 *The $(r+1)$-th lower partial moment about a point c of DMPSD is given by*

$$\mu_{r+1}(t,c) = \frac{g(\theta)}{g'(\theta)} \frac{d\mu_r(t,c)}{d\theta} + \left(\frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} - c \right) \cdot \tag{8}$$
$$\mu_r(t,c) - \alpha(s-c)^r \frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}, \qquad r \geq 0.$$

Proof. For DMPSD with $s < t$ we have

$$\mu_r(t,c) = \sum_{x=0}^{t} (x-c)^r \frac{a(x)[g(\theta)]^x}{f(\theta)} + \left[(s-c)^r - (s+1-c)^r\right] \alpha \frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}. \tag{9}$$

Differentiation (9) with respect to θ gives

$$\frac{d\mu_r(t,c)}{d\theta} = \sum_{x=0}^{t} (x-c)^r a(x) \frac{x[g(\theta)]^{x-1} g'(\theta)}{f(\theta)} - \sum_{x=0}^{t} (x-c)^r a(x) \frac{[g(\theta)]^x f'(\theta)}{[f(\theta)]^2} + \left[(s-c)^r - (s+1-c)^r\right] \alpha a(s+1)[g(\theta)]^{s+1} \left[(s+1)\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right].$$

Moreover, from (9) we have

$$\alpha \sum_{x=0}^{t} (x-c)^r \frac{[g(\theta)]^x a(x)}{f(\theta)} = \mu_r(t,c) - \beta(s-c)^r.$$

Hence

$$\frac{d\mu_r(t,c)}{d\theta} = \frac{g'(\theta)}{g(\theta)} \mu_{r+1}(t,c) + \left[c\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right] \mu_r(t,c) + (s-c)^r \alpha a(s+1) \frac{[g(\theta)]^s g'(\theta)}{f(\theta)}.$$

We get (8) after rearranging above formula. ■

Using (8) with $c = 0$ and next with $c = m_1(t)$ we obtain relations for lower partial ordinary and central moments of DMPSD

$$m_{r+1}(t) = \frac{g(\theta)}{g'(\theta)} \frac{dm_r(t)}{d\theta} + \frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} m_r(t) - \alpha s^r \frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0. \tag{10}$$

$$\nu_{r+1}(t) = \frac{g(\theta)}{g'(\theta)} \frac{d\nu_r(t)}{d\theta} + \alpha a(s+1) \frac{[g(\theta)]^{s+1}}{f(\theta)} \nu_r(t) - \alpha a(s+1) \frac{[g(\theta)]^{s+1}}{f(\theta)} [s - m_1(t)]^r, \quad r \geq 0, \tag{11}$$

where

$$m_1(t) = \frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} + \alpha \frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}. \tag{12}$$

Now we derive recurrence relation for lower partial descending factorial moments.

Theorem 2 *The $(r+1)$-th lower partial descending factorial moment about a point c of DMPSD is given by*

$$\mu_{(r+1)}(t,c) = \frac{g(\theta)}{g'(\theta)}\frac{d\mu_{(r)}(t,c)}{d\theta} + \left(\frac{f'(\theta)}{f(\theta)}\frac{g(\theta)}{g'(\theta)} - c - r\right)\mu_{(r)}(t,c) \tag{13}$$
$$- \alpha(s-c)^{(r)}\frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0.$$

Proof. For DMPSD with $s < t$ we have

$$\mu_{(r)}(t,c) = \sum_{x=0}^{t}(x-c)^{(r)}\frac{[g(\theta)]^x a(x)}{f(\theta)} + \left[(s-c)^{(r)} - (s+1-c)^{(r)}\right] \tag{14}$$
$$\cdot\, \alpha a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}.$$

Differentiation (14) with respect to θ with use of the property of descending factorial

$$x(x-c)^{(r)} = (x-c)^{(r+1)} + (x-c)^{(r)}(c+r) \tag{15}$$

gives

$$\frac{d\mu_{(r)}(t,c)}{d\theta} = \frac{g'(\theta)}{g(\theta)}\sum_{x=0}^{t}(x-c)^{(r+1)}\frac{a(x)[g(\theta)]^x}{[f(\theta)]} \tag{16}$$
$$+ \left[(c+r)\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right]\sum_{x=0}^{t}(x-c)^{(r)}\frac{a(x)[g(\theta)]^x}{[f(\theta)]}$$
$$+ \left[(s-c)^{(r)} - (s+1-c)^{(r)}\right]\alpha a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}$$
$$\cdot \left[(s+1)\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right].$$

Combining (14) and (16) we obtain

$$\frac{d\mu_{(r)}(t,c)}{d\theta} = \frac{g'(\theta)}{g(\theta)}\mu_{(r+1)}(t,c) + \left[(c+r)\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right]\mu_{(r)}(t,c)$$
$$+ \alpha(s-c)^{(r)}a(s+1)\frac{[g(\theta)]^s}{f(\theta)}.$$

Hence we obtain (13). ■

Putting in (13) $c = 0$ we get recurrence relations for lower descending factorial moments of DMPSD

$$m_{(r+1)}(t) = \frac{g(\theta)}{g'(\theta)}\frac{dm_{(r)}(t)}{d\theta} + \left[\frac{f'(\theta)g(\theta)}{g'(\theta)f(\theta)} - r\right]m_{(r)}(t) \tag{17}$$
$$- \alpha s^{(r)}a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0.$$

We end this Section with lower partial ascending factorial moments.

Theorem 3 *The $(r+1)$-th lower partial ascending factorial moment about a point c of DMPSD is given by*

$$\mu_{[r+1]}(t,c) = \frac{g(\theta)}{g'(\theta)}\frac{d\mu_{[r]}(t,c)}{d\theta} + \left[\frac{f'(\theta)}{f(\theta)}\frac{g(\theta)}{g'(\theta)} - c + r\right]\mu_{[r]}(t,c) \tag{18}$$
$$- \alpha(s-c)^{[r]}a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0.$$

Proof. For DMPSD with $s < t$ we have

$$\mu_{[r]}(t,c) = \sum_{x=0}^{t}(x-c)^{[r]}\frac{a(x)[g(\theta)]^x}{f(\theta)} + \left[(s-c)^{[r]} - (s+1-c)^{[r]}\right] \tag{19}$$
$$\cdot\, \alpha a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}.$$

Differentiating (19) with respect to θ and using the property of ascending factorial

$$x(x-c)^{[r]} = (x-c)^{[r+1]} - (x-c)^{[r]}(c-r)$$

we get

$$\frac{d\mu_{[r]}(t,c)}{d\theta} = \frac{g'(\theta)}{g(\theta)}\sum_{x=0}^{t}(x-c)^{[r+1]}\frac{a(x)[g(\theta)]^x}{[f(\theta)]} \tag{20}$$
$$+ \left[(c-r)\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right]\sum_{x=0}^{t}(x-c)^{[r]}\frac{a(x)[g(\theta)]^x}{[f(\theta)]}$$
$$+ \left[(s-c)^{[r]} - (s+1-c)^{[r]}\right]\alpha a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}$$
$$\cdot \left[(s+1)\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right].$$

Combining (19) and (20) gives

$$\frac{d\mu_{[r]}(t,c)}{d\theta} = \frac{g'(\theta)}{g(\theta)}\mu_{[r+1]}(t,c) + \left[(c-r)\frac{g'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)}\right]\mu_{[r]}(t,c)$$
$$+ \alpha(s-c)^{[r]}a(s+1)\frac{[g(\theta)]^s}{f(\theta)}.$$

After some simply calculations we get (18). ■

Putting in (18) $c = 0$ we get recurrence relations for lower ascending factorial moments of DMPSD The $(r+1)$-th lower partial ascending factorial moment of

DMPSD is given by

$$m_{[r+1]}(t) = \frac{g(\theta)}{g'(\theta)} \frac{dm_{[r]}(t)}{d\theta} + \left(\frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} + r \right) m_{[r]}(t) \\ - \alpha s^{[r]} a(s+1) \frac{[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0. \tag{21}$$

Remark 1 *From (8) and (17) for $t \to \infty$ we can get relations for complete moments of non-zero DMPSD obtained by Murat and Szynal (2003).*

4 Recurrence relations for upper partial moments of DMPSD

It is obvious that for $t \geq s$ the upper partial moments are independent of a deformation point s and recurrence relations for these moments are the same as formulae obtained by Gupta (1974). So we consider only the situation when s is greater than t. Following the methods and techniques from previous section we obtain undermentioned results.

Theorem 4 *The $(r+1)$-th upper partial moment about a point c of DMPSD is given by*

$$\mu^{r+1}(t,c) = \frac{g(\theta)}{g'(\theta)} \frac{d\mu^r(t,c)}{d\theta} + \left(\frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} - c \right) \mu^r(t,c) \\ - \alpha (s-c)^r \frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0. \tag{22}$$

Using (22) with $c = 0$ and next with $c = m_1(t)$ we obtain relations for upper partial ordinary and central moments of DMPSD.

$$m^{r+1}(t) = \frac{g(\theta)}{g'(\theta)} \frac{dm^r(t)}{d\theta} + \frac{f'(\theta)}{f(\theta)} \frac{g(\theta)}{g'(\theta)} m^r(t) \\ - \alpha s^r \frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0, \tag{23}$$

$$\nu^{r+1}(t) = \frac{g(\theta)}{g'(\theta)} \frac{d\nu^r(t)}{d\theta} + \alpha a(s+1) \frac{[g(\theta)]^{s+1}}{f(\theta)} \nu^r(t) \\ - \alpha a(s+1) \frac{[g(\theta)]^{s+1}}{f(\theta)} [s - m_1(t)]^r, \quad r \geq 0, \tag{24}$$

where $m_1(t)$ is given by (12).

Theorem 5 *The $(r+1)$-th upper partial descending factorial moment about a point c of DMPSD is given by*

$$\mu^{(r+1)}(t,c) = \frac{g(\theta)}{g'(\theta)}\frac{d\mu^{(r)}(t,c)}{d\theta} + \left(\frac{f'(\theta)}{f(\theta)}\frac{g(\theta)}{g'(\theta)} - c - r\right)\mu^{(r)}(t,c) \tag{25}$$
$$- \alpha(s-c)^{(r)}\frac{a(s+1)[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0.$$

From (25) for $c = 0$ we obtain recurrence relations for upper descending factorial partial moments of DMPSD

$$m^{(r+1)}(t) = \frac{g(\theta)}{g'(\theta)}\frac{dm^{(r)}(t)}{d\theta} + \left[\frac{f'(\theta)g(\theta)}{g'(\theta)f(\theta)} - r\right]m^{(r)}(t) \tag{26}$$
$$- \alpha s^{(r)}a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0.$$

Theorem 6 *The $(r+1)$-th upper partial ascending factorial moment about a point c of DMPSD is given by*

$$\mu^{[r+1]}(t,c) = \frac{g(\theta)}{g'(\theta)}\frac{d\mu^{[r]}(t,c)}{d\theta} + \left[\frac{f'(\theta)}{f(\theta)}\frac{g(\theta)}{g'(\theta)} - c + r\right]\mu^{[r]}(t,c) \tag{27}$$
$$- \alpha(s-c)^{[r]}a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0.$$

From (27) with $c = 0$ we get

$$m^{[r+1]}(t) = \frac{g(\theta)}{g'(\theta)}\frac{dm^{[r]}(t)}{d\theta} + \left(\frac{f'(\theta)}{f(\theta)}\frac{g(\theta)}{g'(\theta)} + r\right)m^{[r]}(t) \tag{28}$$
$$- \alpha s^{[r]}a(s+1)\frac{[g(\theta)]^{s+1}}{f(\theta)}, \quad r \geq 0.$$

Remark 2 *From (22) and (26) for $t \to 0$ we can get relations for complete moments of DMPSD obtained by Murat and Szynal (2003).*

5 Examples

This Section is devoted to illustrative examples of formulae which we obtained in Section 3 and 4. We consider some special cases of DMPSD.

Example 1 *A discrete random variable X has* **deformed generalized Poisson distribution** *if its probability mass function is given by (1) with*

$$a(x) = \frac{b(b+ax)^{x-1}}{x!},\ g(\theta) = \theta e^{-a\theta},\ f(\theta) = e^{b\theta},$$

where $x = 0,1,2,...;\ \theta > 0,\ |\ \theta a\ | < 1$.

The generalized Poisson distribution ($\alpha = 0$) *was introduced in Consul and Jain (1973), and studied extensively by Consul (1989). Applications of the generalized Poisson distribution can be found in settings where one seeks to describe the distribution of an event that occurs rarely in a short period, but where we observe the frequency of its occurrence in longer periods of time. The distribution was found to accurately describe phenomena as diverse as the observed number of industrial accidents and injuries, where a learning effect may be present, the spatial distribution of insects, where initial occupation of a spot by a member of the species has an influence on the attractiveness of the spot to other members of the species, and the number of units of different commodities purchased by consumers, where current sales have an impact on the level of subsequent sales through repeat purchases.*

Using (10) we get the following recurrence relation for lower partial ordinary moments for the deformed generalized Poisson distribution

$$m_{r+1}(t) = \frac{\theta}{1-a\theta}\frac{dm_r(t)}{d\theta} + \frac{b\theta}{1-a\theta}m_r(t) - \alpha s^r \frac{(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)},\ r \geq 0.$$

From (11) we obtain the recurrence relation for lower partial central moments for the deformed generalized Poisson distribution as follows

$$\nu_{r+1}(t) = \frac{\theta}{1-a\theta}\frac{d\nu_r(t)}{d\theta} + \alpha\frac{b(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)}\nu_r(t) - \alpha\frac{b(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)}\left[s^r - m_1(t)\right]^r,\ r \geq 0,$$

where $m_1(t) = \dfrac{b\theta}{1-a\theta} - \alpha\dfrac{b(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)}$.

Relations for partial factorial moments for deformed generalized Poisson distribution we obtain from (17), (21), (26) and (28)

$$m_{(r+1)}(t) = \frac{\theta}{1-a\theta}\frac{dm_{(r)}(t)}{d\theta} + \left(\frac{b\theta}{1-a\theta} - r\right)m_{(r)}(t) - \alpha s^{(r)}\frac{(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)},\ r \geq 0,$$

$$m_{[r+1]}(t) = \frac{\theta}{1-a\theta}\frac{dm_{[r]}(t)}{d\theta} + \left(\frac{b\theta}{1-a\theta} + r\right)m_{[r]}(t) - \alpha s^{[r]}\frac{(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)},\ r \geq 0,$$

$$m^{(r+1)}(t) = \frac{\theta}{1-a\theta}\frac{dm^{(r)}(t)}{d\theta} + \left(\frac{b\theta}{1-a\theta} - r\right) m^{(r)}(t) - \alpha s^{(r)} \frac{(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)},\ r \geq 0,$$

$$m^{[r+1]}(t) = \frac{\theta}{1-a\theta}\frac{dm^{[r]}(t)}{d\theta} + \left(\frac{b\theta}{1-a\theta} + r\right) m^{[r]}(t) - \alpha s^{[r]} \frac{(b+a+as)^s}{(s+1)!}\theta^{s+1}e^{-\theta(b+a+as)},\ r \geq 0.$$

From above formulae putting $a = 0$ and $b = 1$ we can obtain recurrence relation for partial moments of deformed Poisson distribution with parameter θ.

Example 2 *A random variable X has* **deformed generalized negative binomial distribution** *if its probability mass function is given by (1) with*

$$a(x) = \frac{n\Gamma(n+bx)}{x!\Gamma(n+bx-x+1)},\ g(\theta) = \theta(1-\theta)^{b-1},\ f(\theta) = (1-\theta)^{-n},$$

where for $x = 0,1,2,...;\ 0 < \theta < 1,\ |\ \theta b\ | < 1$.

The generalized negative binomial distribution ($\alpha = 0$) was obtained as a particular family of Lagrangian distributions by Consul and Shenton (1972, 1973). They showed that the class of Lagrangian distributions provide the distributions of the busy periods of a single server under certain conditions. Neyman (1939, 1966) showed that the distribution of the total number N of infected beings starting from those infected by a single infectious and up to the moment of extinction of the epidemic is a particular case of the generalized negative binomial distribution. Kumar (1981) gave some more applications of the class of the generalized negative binomial distribution in queueing theory and epidemiology.

Using (10) we get the following recurrence relation for lower partial ordinary and central moments for the deformed negative binomial distribution

$$m_{r+1}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_r(t)}{d\theta} + \frac{n\theta}{1-b\theta}m_r(t) - \alpha s^r \frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^{s+1}}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}},\ r \geq 0,$$

$$\nu_{r+1}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{d\nu_r(t)}{d\theta} + \frac{n\theta}{1-b\theta}\nu_r(t) - \alpha s^r \frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^{s+1}}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}}\left[s - m_1(t)\right]^r,\ r \geq 0,$$

where $m_1(t) = \dfrac{n\theta}{1-\theta b} - \alpha\dfrac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^{s+1}}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}}.$

Relations for lower partial descending and ascending factorial moments for deformed generalized negative binomial distribution we obtain from (17) and (21)

$$m_{(r+1)}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_{(r)}(t)}{d\theta} + \left(\frac{n\theta}{1-b\theta} - r\right) m_{(r)}(t) - \alpha s^{(r)} \frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^{s+1}}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}},\ r \geq 0,$$

$$m_{[r+1]}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_{[r]}(t)}{d\theta} + \left(\frac{n\theta}{1-b\theta} + r\right) m_{[r]}(t) - \alpha s^{[r]} \frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^{s+1}}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}},\ r \geq 0,$$

$$m^{(r+1)}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm^{(r)}(t)}{d\theta} + \left(\frac{n\theta}{1-b\theta} - r\right) m^{(r)}(t) - \alpha s^{(r)} \frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^{s+1}}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}},\ r \geq 0,$$

$$m^{[r+1]}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm^{[r]}(t)}{d\theta} + \left(\frac{n\theta}{1-b\theta} + r\right) m^{[r]}(t) - \alpha s^{[r]} \frac{n\Gamma(n+bx)[\theta(1-\theta)^{b-1}]^{s+1}}{x!\Gamma(n+bx-x+1)(1-\theta)^{-n}},\ r \geq 0.$$

Putting $b = 0$ we can get relations for partial moments of deformed binomial distribution and putting $b = 1$ we can obtain relations for partial moments of deformed binomial.

Example 3 *A random variable X has* **deformed generalized logarithmic series distribution** *if its probability mass function fulfil (1) with*

$$a(x) = \frac{\Gamma(bx)}{x\Gamma(x)\Gamma(bx-x+1)},\ g(\theta) = \theta(1-\theta)^{b-1},\ f(\theta) = -ln(1-\theta).$$

The generalized logarithmic series distribution ($\alpha = 0$) was obtained by Jain and Gupta (1973) through Lagrange expansion of the ordinary logarithmic series distribution. It is also possible to get the generalized logarithmic series distribution as a limiting form of the zero truncated generalized negative binomial distribution, see Jain (1975).

Relations for partial ordinary and central moments for deformed generalized logarithmic series distribution are get from (10), (17) and (21)

$$m_{r+1}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_r(t)}{d\theta} - \frac{\theta}{(1-b\theta)\ln(1-\theta)}m_r(t) + \alpha s^r \frac{\Gamma(b+bs)\left[\theta(1-b\theta)^{b-1}\right]^{s+1}}{(s+1)\Gamma(s+1)\Gamma(b+bs-s)\ln(1-\theta)},\ r \geq 0,$$

$$\nu_{r+1}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{d\nu_r(t)}{d\theta} - \frac{\theta}{(1-b\theta)\ln(1-\theta)}\nu_r(t) + \alpha s^r \frac{\Gamma(b+bs)\left[\theta(1-b\theta)^{b-1}\right]^{s+1}}{(s+1)\Gamma(s+1)\Gamma(b+bs-s)\ln(1-\theta)}\Big(s-m_1(t)\Big)^r,\ r \geq 0,$$

where

$$m_1(t) = -\frac{\theta}{(1-b\theta)\ln(1-\theta)} + \alpha s^r \frac{\Gamma(b+bs)\left[\theta(1-b\theta)^{b-1}\right]^{s+1}}{(s+1)\Gamma(s+1)\Gamma(b+bs-s)\ln(1-\theta)}.$$

Using formulae (17) and (21) we obtain partial factorial moments as follows

$$m_{(r+1)}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_{(r)}(t)}{d\theta} - \left(\frac{\theta}{(1-b\theta)\ln(1-\theta)} + r\right)m_{(r)}(t) + \alpha s^r \frac{\Gamma(b+bs)\left[\theta(1-b\theta)^{b-1}\right]^{s+1}}{(s+1)\Gamma(s+1)\Gamma(b+bs-s)\ln(1-\theta)},\ r \geq 0,$$

$$m_{[r+1]}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm_{[r]}(t)}{d\theta} - \left(\frac{\theta}{(1-b\theta)\ln(1-\theta)} - r\right)m_{[r]}(t) + \alpha s^r \frac{\Gamma(b+bs)\left[\theta(1-b\theta)^{b-1}\right]^{s+1}}{(s+1)\Gamma(s+1)\Gamma(b+bs-s)\ln(1-\theta)},\ r \geq 0,$$

$$m^{(r+1)}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm^{(r)}(t)}{d\theta} - \left(\frac{\theta}{(1-b\theta)\ln(1-\theta)} + r\right)m^{(r)}(t) + \alpha s^r \frac{\Gamma(b+bs)\left[\theta(1-b\theta)^{b-1}\right]^{s+1}}{(s+1)\Gamma(s+1)\Gamma(b+bs-s)\ln(1-\theta)},\ r \geq 0,$$

$$m^{[r+1]}(t) = \frac{\theta(1-\theta)}{1-b\theta}\frac{dm^{[r]}(t)}{d\theta} - \left(\frac{\theta}{(1-b\theta)\ln(1-\theta)} - r\right)m^{[r]}(t) + \alpha s^r \frac{\Gamma(b+bs)\left[\theta(1-b\theta)^{b-1}\right]^{s+1}}{(s+1)\Gamma(s+1)\Gamma(b+bs-s)\ln(1-\theta)},\ r \geq 0.$$

When we put $b = 1$ in above formulae we get partial moments of deformed Fisher's logarithmic series distribution.

Example 4 *A random variable X has* **deformed lost games distribution** *if its probability mass function is defined by (1) with*

$$a(x)=\frac{a}{2x-a}\binom{2x-a}{x},\ f(\theta)=\theta^a,\ g(\theta)=\theta(1-\theta),$$

where $x=a,a+1,...;\ a\geq 1,\ 0<\theta<\frac{1}{2}$.

The lost games distribution ($\alpha=0$) *was obtained by Kemp and Kemp (1968) as the distribution of the total number of games lost by the ruined gambler starting with a monetary units against an infinitely rich adversary. They also showed that this distribution can be obtained as the distribution of the number of customers served in a busy period starting with a customers of an M/M/1 queue.*

In this case relations for partial ordinary and central moments we get from (10), (17) and (21)

$$m_{r+1}(t)=\frac{\theta(1-\theta)}{1-2\theta}\frac{dm_r(t)}{d\theta}+\frac{a(1-\theta)}{1-2\theta}m_r(t)$$
$$-\alpha s^r\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\theta^{s+1-a}(1-\theta)^{s+1},\ r\geq 0,$$

$$\nu_{r+1}(t)=\frac{\theta(1-\theta)}{1-2\theta}\frac{d\nu_r(t)}{d\theta}+\frac{a(1-\theta)}{1-2\theta}\nu_r(t)$$
$$-\alpha s^r\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\theta^{s+1-a}(1-\theta)^{s+1}\Big(s-m_1(t)\Big)^r,\ r\geq 0,$$

where $m_1(t)=\frac{a(1-\theta)}{1-2\theta}-\alpha\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\theta^{s+1-a}(1-\theta)^{s+1}$.

Relations for partial factorial moments are as follows

$$m_{(r+1)}(t)=\frac{\theta(1-\theta)}{1-2\theta}\frac{dm_{(r)}(t)}{d\theta}+\left(\frac{a(1-\theta)}{1-2\theta}-r\right)m_{(r)}(t)$$
$$-\alpha s^{(r)}\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\theta^{s+1-a}(1-\theta)^{s+1},\ r\geq 0,$$

$$m_{[r+1]}(t)=\frac{\theta(1-\theta)}{1-2\theta}\frac{dm_{[r]}(t)}{d\theta}+\left(\frac{a(1-\theta)}{1-2\theta}-r\right)m_{[r]}(t)$$
$$-\alpha s^{[r]}\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\theta^{s+1-a}(1-\theta)^{s+1},\ r\geq 0,$$

$$m^{(r+1)}(t)=\frac{\theta(1-\theta)}{1-2\theta}\frac{dm^{(r)}(t)}{d\theta}+\left(\frac{a(1-\theta)}{1-2\theta}-r\right)m^{(r)}(t)$$
$$-\alpha s^{(r)}\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\theta^{s+1-a}(1-\theta)^{s+1},\ r\geq 0,$$

$$m^{[r+1]}(t) = \frac{\theta(1-\theta)}{1-2\theta}\frac{dm^{[r]}(t)}{d\theta} + \left(\frac{a(1-\theta)}{1-2\theta} - r\right) m^{[r]}(t) - \alpha s^{[r]} \frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\theta^{s+1-a}(1-\theta)^{s+1},\ r \geq 0.$$

Example 5 *A random variable X has* **deformed distribution of the number of customers served in a busy period of the queue M/M/1** *if its probability mass function is given by (1) with*

$$a(x) = \frac{a}{2x-a}\binom{2x-a}{x},\ f(\theta) = \left(\frac{1+\theta}{\theta}\right)^a,\ g(\theta) = \frac{\theta}{(1+\theta)^2}.$$

In this case partial ordinary and central moments fulfil recurrence relations

$$m_{r+1}(t) = \frac{\theta(1+\theta)}{1-\theta}\frac{dm_r(t)}{d\theta} + \frac{a}{\theta-1}m_r(t) - \alpha s^r \frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\frac{\theta^{a+s+1}}{(1+\theta)^{a+2s+2}},\ r \geq 0,$$

$$\nu_{r+1}(t) = \frac{\theta(1+\theta)}{1-\theta}\frac{d\nu_r(t)}{d\theta} + \frac{a}{\theta-1}\nu_r(t) - \alpha \frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\frac{\theta^{a+s+1}}{(1+\theta)^{a+2s+2}}\Big(s - m_1(t)\Big)^r,\ r \geq 0,$$

where

$$m_1(t) = \frac{a}{\theta-1} - \alpha\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\frac{\theta^{a+s+1}}{(1+\theta)^{a+2s+2}}.$$

Relations for partial factorial moments are as follows

$$m_{(r+1)}(t) = \frac{\theta(1+\theta)}{1-\theta}\frac{dm_{(r)}(t)}{d\theta} + \left(\frac{a}{\theta-1} - r\right) m_{(r)}(t) - \alpha s^{(r)} \frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\frac{\theta^{a+s+1}}{(1+\theta)^{a+2s+2}},\ r \geq 0,$$

$$m_{[r+1]}(t) = \frac{\theta(1+\theta)}{1-\theta}\frac{dm_{[r]}(t)}{d\theta} + \left(\frac{a}{\theta-1} + r\right) m_{[r]}(t) - \alpha s^{[r]} \frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\frac{\theta^{a+s+1}}{(1+\theta)^{a+2s+2}},\ r \geq 0,$$

$$m^{(r+1)}(t) = \frac{\theta(1+\theta)}{1-\theta}\frac{dm^{(r)}(t)}{d\theta} + \left(\frac{a}{\theta-1} - r\right) m^{(r)}(t) - \alpha s^{(r)} \frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\frac{\theta^{a+s+1}}{(1+\theta)^{a+2s+2}},\ r \geq 0,$$

$$m^{[r+1]}(t) = \frac{\theta(1+\theta)}{1-\theta}\frac{dm^{[r]}(t)}{d\theta} + \left(\frac{a}{\theta-1}+r\right)m^{[r]}(t) - \alpha s^{[r]}\frac{a}{2s-a+2}\binom{2s-a+2}{s+1}\frac{\theta^{a+s+1}}{(1+\theta)^{a+2s+2}},\ r \geq 0.$$

6 Conclusion

Obtained recurrence relations for partial lower and upper moments generalize and extend formulae for moments established by Gupta, Gupta and Thripati (1986). They also complement formulae given by Murat and Szynal (2003).

References

Antle J. M., (2010), *Asymmetry, Partial Moments, and Production Risk*, American Journal of Agricultural Economics, 92(5), 1294–1309.

Bortkiewicz L. von, (1898), *Das Gesetz der Kleinen Zahlen*, Teubner, Leipzig.

Cohen A. C., (1960), *Estimation in the Poisson distribution when sample values of* $c+1$ *are sometimes erroneously reported as c.* Ann. Inst. Stat. Math. 9, 189–193.

Consul P. C., (1989), *Generalized Poisson Distributions: Properties and Applications*, Volume 99 of Statistics: Textbooks and Monographs, Marcel Dekker Inc., New York.

Consul P. C., Jain G. C., (1973), *A generalization of the Poisson distribution*, Technometrics 15(4), 791–799.

Consul P. C., Shenton L. R., (1972), *Use of Lagrangian expansion for generating discrete for generating discrete generalized probability distributions*, SIAM J. Appl. Math. 98, 239–248.

Consul P. C., Shenton L. R., (1973), *Some interesting properties of Lagrangian distributions*, Communication in Statistics, 2 (3), 263–272.

Gupta R.C., (1974), *Modified power series distributions and some of its applications*, Sankhy$\tilde{a}$, ser. B, vol. 35, 288–298.

Gupta P. L., Gupta R. C. and Tripathi R. C., (1986), *Incomplete moments modified power series distributions with applications*, Communications in Statistics – Theory and Methods, vol. 15(3), 999–1015.

Hassan A., Ahmad P. B., (2009), *Misclassified size-biased modified power series distribution and its applications*, Mathematica Bohemica, Vol. 134, No. 1, 1–17.

Jain G.C., Gupta R.P., (1973), *A logarithmic type distribution*, Trabajos Estadist. 24, 99–105.

Jain G.C., (1975), *On power series distributions associated with Lagrange expansion*, Biometrische Zeitschrift Bd. 17, 85–97.

Kemp A. W., Kemp C. D., (1968), *On a distribution associated with certain stochastic processes*, J. Roy. Statist. Soc., Ser. B, 30, 160–163.

Kumar A., (1981), *Some application of Lagrangian distributions in queueing theory and epidemiology*, Commun. Statist.-Theor. Meth. A 10 (14), 1429–1436.

Murat M., Szynal D., (2003), *Moments of certain deformed probability distributions*, Commun. Statist.-Theory Meth., vol. 32, no. 2, 291–314.

Neyman J., (1939), *On a new class of contagious distributions applicable in entomology and bacteriology*, Ann. Math. Statist. 10, 35–67.

Neyman, J., (1966), *Certain chance mechanisms involving discrete distributions*, Ed. G. P. Patil, Stat. Pub. SOC., Calcutta.

Williford W. O., Bingham S. F., (1979), *Bayesian estimation of the parameters in two modified Poisson distributions*, Commun. Statist.-Theor. Meth. A8(13), 1315–1326.

Winkler R. L., Roodman G. M., Britney R. R., (1972), *The Determination of Partial Moments*, Management Science 19(3), 290–296.

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Investment risks and their measurement

Keywords: risk measurement, asset classes, cryptocurrency, CRIX

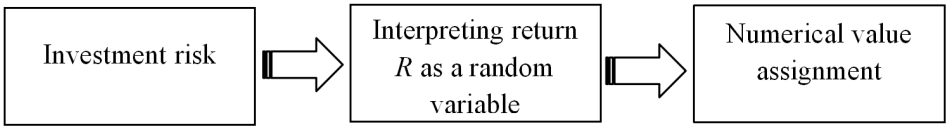
Abstract

The article is devoted to analysis of investment risks and their measurement. Three approaches for risk measurement are examined. These approaches have been applied to risk estimation of basic cryptocurrencies. Statistical assessment of basic risk measures from each approach was accomplished. The investigation shows that cryptocurrencies have completely distinctive characteristics of risk-return corresponding. It distinguishes cryptocurrencies from traditional investment assets and from new investment opportunities. The results are important for investment and risk management purposes.

1 Introduction

Risk takes a central place in the framework of investment decision making. The relationship between return and risk is in the core of modern investment thinking. As a rule, higher return should be associated with higher risk. The opposite correspondence is also true: higher risk should be covered by additional return (risk premium). Consequently, it is very logically to analyze risk-return correspondence before investment decision.

A source anchor of construction of such correspondence is risk measurement. Risk measurement is a procedure of assigning some numerical value to risk. This procedure can be formalized for investment risk by following scheme:



Procedure of risk measurement

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So, risk measurement supposes to introduce some mapping ρ which each random variable R (representing return of investment asset) assigned nonnegative number $\rho(R) \in [0; +\infty]$. Let us consider this procedure in details. The return of investment over a period of time $[t; t+1]$ will be expressed through the formula:

$$R_{t,\,t+1} = \frac{P_{t+1} - P_t}{P_t},$$

where P_t and P_{t+1} are prices of an investment asset at times t and $t+1$, respectively. $R_{t,\,t+1}$ will be a random variable, because the future price P_{t+1} is unknown. Thereafter R, which reflects return over the time, is also a random variable.

Assigning a numerical value for risk is complicated because various approaches for presentation of mapping ρ exist. Three conceptual approaches are the most significant ones:

- Risk measurement is based on reflecting the variability of return and income.
- Risk measurement is focused on losses in negative situation.
- Risk measurement associates with sensitivity of return to some factors. Measurement is focused on response level.

Each approach incorporates some important characteristics of multifaceted notion of risk and has a number of indicators. In general, there are several dozen of risk measures, which represent one or another aspect of risk (example is presented in Szego (2004)). An attempt of understanding the essence of properties which should be represented in risk measure was formulated in Artzner et al. (1999). The authors created a notion of a coherent risk measure. Risk measure ρ is coherent if satisfies the following properties (axioms):

Axiom 1 Sub-additivity. For all R_1 and R_2 we have

$$\rho(R_1 + R_2) \leq \rho(R_1) + \rho(R_2)$$

Axiom 2 Positive homogeneity. For all R and for all $\lambda \geq 0$ we have

$$\rho(\lambda R) = \lambda \rho(R)$$

Axiom 3 Monotonicity. If $R_1 \geq R_2$ then $\rho(R_1) \leq \rho(R_2)$

Axiom 4 Translation invariance. For all R and for all $\alpha \geq 0$ we have

$$\rho(R + \alpha) = \rho(R) - \alpha$$

Each of these axioms formalizes some essential investment risk property. Thus Axiom 1 presents diversification effect. Axiom 2 describes linear increase of risk if some investment position is linearly increased. Axiom 3 presents a natural property: if returns for one investment are always higher than returns for other investment, then risk of the first investment is lower. Axiom 4 formalizes adding to investment a free-risk asset.

Examples of coherent risk measures are Conditional Value-at-Risk (considered below, see Rockafellar and Uryasev (2000)) and the Fischer measure (see Fischer (2003)).

It is necessary to note that presented approach for coherency is not unique. Approaches of coherency are considered in Kaminskyi (2006).

Below we consider applications of three approaches of risk measurement to cryptocurrencies. Cryptocurrencies are one of the alternative investment assets which demonstrated high developing since last years. The investment problems of cryptocurrencies are discussed in Lee, Guo and Wang (2018), Chan et al. (2017), Gangwal (2018) and Trimborn, Mingyang and Härdle (2017),

We have chosen for analysis cryptocurrencies with capitalization higher than 1 billion USD. They are:

Table 1. A list of cryptocurrencies chosen for analysis

Cryptocurrency	**Ticker tape**	**Start day of trading**	**Capitalisation on 17.08.2018**	**Share of total market capitalization**
Bitcoin	BTC	18.07.2010	$111.23B	52.1%
Ethereum	ETH	10.03.2016	$30.27B	14.2%
Ripple	XRP	22.01.2015	$12.28B	5.8%
Bitcoin Cash	BCH	03.08.2017	$9.31B	4.4%
EOS	EOS	02.07.2017	$4.34B	2.0%
Stellar Lumens	XLM	22.02.2017	$4.20B	2.0%
Litecoin	LTC	24.08.2016	$3.31B	1.6%
Tether	USDT	14.04.2017	$2.68B	1.3%
Cardano	ADA	31.12.2017	$2.59B	1.2%
Monero	XMR	30.01.2015	$1.55B	0.7%
Ethereum Classic	ETC	28.07.2016	$1.42B	0.7%
TRON	TRX	14.11.2017	$1.39B	0.7%
IOTA	MIOTA	14.06.2017	$1.34B	0.6%
Dash	DASH	04.03.2017	$1.29B	0.6%
NEO	NEO	08.09.2017	$1.14B	0.5%

Source: Data sources used for investigation were https://investing.com and http://thecrix.de

2 The variability approach for risk measurement

The variability approach is focused on dispersion or deviation from an expected outcome. The most simple risk measure is a range which equals to difference between maximum and minimum possible values:

$$L(R) = \max_{[0,T]} R(t) - \min_{[0,T]} R(t).$$

This risk indicator is important for the investor from the point of view of receiving a general picture about future possibilities (it is assumed that future distribution will be the same as historical distribution). The shortcoming of this risk indicator is that maximum and minimum prices were on peak and crisis times. These may be rare events and not relevant for periods of stability. Consequently, it is more efficient to use inter-quartile range:

$$Q(R) = Q_{75\%}(R(t)) - Q_{25\%}(R(t))$$

Of course, the most famous risk measure used in this approach is standard deviation which characterizes deviation from the expected value of R:

$$\sigma(R) = \sqrt{\int_{-\infty}^{+\infty} (R - E(R))^2 \, \mathrm{dF}(R)}$$

Expected value of R is defined as

$$E(R) = \int_{-\infty}^{+\infty} R dF(R)$$

where F is the distribution function of the random variable R.

If we use statistical estimations of R, then unbiased estimate of standard deviation is:

$$\hat{\sigma}(R) = \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} (R(t) - E(R))^2}$$

Statistical estimation of $E(R)$ can be calculated by formula:

$$E(R) = \frac{1}{T} \sum_{t=1}^{T} R(t) \ (T\text{-number of periods}).$$

The other indicators which can be used for risk measurement in the frameworks of the variability approach are skewness and kurtosis. Skewness summarizes divergence from symmetry of distribution:

$$S(R) = E\left(\frac{R - E(R)}{\sigma(R)}\right)^3 = \frac{\mu_3(R)}{\sigma(R)^3}$$

where $\mu_3(R) = E(R - E(R))^3$.
The unbiased statistical estimation of skewness is:

$$\hat{S}(R)_{\text{Unbiased}} = \frac{\sqrt{(T-1)T}}{T-2} \cdot \frac{\frac{1}{T}\sum_{t=1}^{T}(R(t) - E(R))^3}{\left(\frac{1}{T}\sum_{t=1}^{T}(R(t) - E(R))^2\right)^{(3/2)}}.$$

Negative skewness indicates a long left tail of distribution, or the possibility of larger losses than profits. Positive skewness is a desirable characteristic for risk-averse investors. The motivation of that is based on the expected utility theory. Typically, the third derivative of the utility function of a risk-averse investor is positive (see e.g. Scott and Horvath (1980)) and this derivative is a multiplier for skewness in the Taylor expansion of expected utility.

The kurtosis (sometimes the term "excess kurtosis" is used) coefficient K can be considered as assessment of the size of distribution tails:

$$K(R) = E\left(\frac{R - E(R)}{\sigma(R)}\right)^4 - 3 = \frac{\mu_4(R)}{\sigma(R)^4} - 3$$

where $\mu_4(R) = E(R - E(R))^4$.

Kurtosis can be considered as measure of risk associated with heavy tails or outliers. Kurtosis greater than 0 indicates a fatter tail than the normal distribution has. Hence, this distribution may generate more extreme values which lead to potential catastrophic risks. The sample kurtosis is

$$\hat{K}(R) = \frac{\frac{1}{T}\sum_{t=1}^{T}(R(t) - E(R))^4}{\left(\frac{1}{T}\sum_{t=1}^{T}(R(t) - E(R))^2\right)^2} - 3$$

An unbiased estimator of the sample excess kurtosis is

$$\hat{K}(R)_{\text{Unbiased}} = \frac{(T-1)}{(T-2)(T-3)} \cdot \left((T+1)\hat{K}(R) + 6\right)$$

Results of statistical estimations for considered risk measures are presented at the Table 2.

Risk-return correspondence at the frameworks of classical consideration expected return and standard deviation (Markowitz (1959)) is presented on Figure 1.

Table 2. Statistical estimations of indicators from variability approach for risk measurement (daily return, time period: 01.01.2018 – 17.08.2018)

Crypto-currency	Expected return	Range	Inter-quartile	Standard deviation	Skew-ness	Kurt-osis
BTC	-0.22%	31.00%	3.20%	4.70%	-0.16	1.15
ETH	-0.23%	34.80%	3.70%	5.69%	-0.10	0.66
XRP	-0.58%	54.50%	4.60%	6.69%	0.14	2.43
BCH	-0.40%	60.60%	4.90%	7.05%	0.39	3.23
EOS	0.17%	64.10%	4.80%	8.91%	1.23	4.75
XLM	0.15%	86.80%	5.00%	8.66%	1.60	10.07
LTC	-0.41%	52.60%	3.90%	6.24%	0.94	4.82
USDT	-0.01%	4.70%	0.10%	0.44%	0.58	9.72
ADA	-0.54%	63.70%	4.60%	8.11%	1.19	4.57
XMR	-0.33%	45.80%	4.90%	6.56%	-0.07	1.18
ETC	-0.03%	53.20%	4.70%	7.16%	-0.06	1.59
TRX	0.31%	142.60%	4.80%	12.32%	3.86	31.50
MIOTA	-0.58%	52.70%	5.40%	7.36%	-0.06	0.63
DASH	-0.64%	37.20%	4.10%	5.79%	0.07	0.92
NEO	-0.16%	92.60%	5.60%	9.93%	1.70	9.11

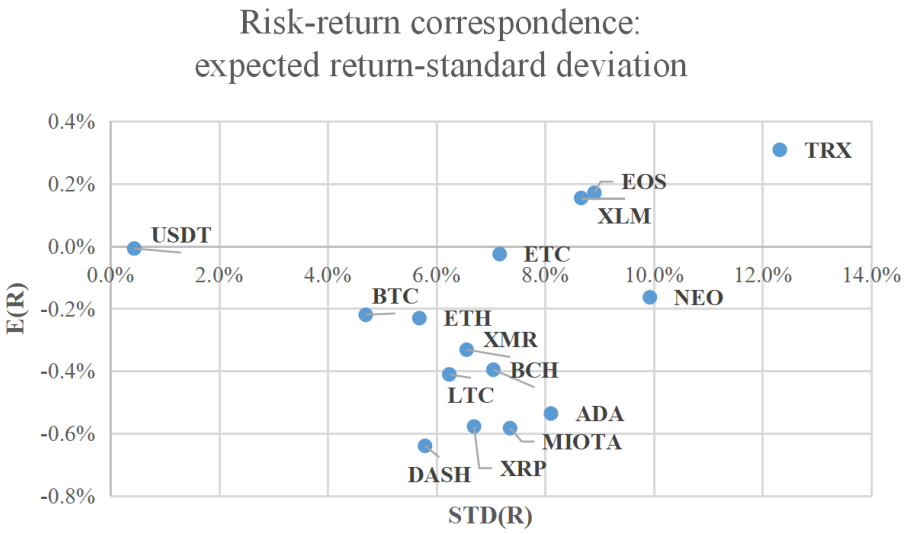

Figure 1. Correspondence "expected return – standard deviation"

So, Figure 1 illustrates an interesting property of risk-return correspondence for cryptocurrencies: a transparent dependency between risk and return is absent.

3 Risk measurement as losses in negative situation

This conceptual approach is based on considering different measures relating to the interpretation of "negative situation" for the investor. Among others, it is possible to mark out downside deviation risk measure. This measure focuses on the returns that below MAR (minimum acceptable return). MAR should be considered as a minimum threshold. Another risk measure at analysing frameworks is TUW (time under the water). This measure calculates how long does the investor wait to recover its money at the start of the drown down period. But, of course, the most popular in this group is the left-tail risk measures, such as Value-at-Risk (VaR) (Holton 2003). This risk measure presents a quantile corresponding to some level of safety (example 95%, 99% or 99.9%). The economic logic of VaR is based on risk covering. If, for example, VaR orients for 95%, then 5% biggest losses will throw off. VaR will cover maximum losses at the framework of 95% possibilities.

VaR is a very efficient measure for market risk. Moreover, it is a regulative risk measure in banking. But together with advantages this measure has shortcomings, too. First shortcoming raises from the fact that VaR is really only one point of probability distribution function (pdf). Behaviour of pdf left-side and right-side from VaR is out of consideration. Second gap of VaR is absence of coherency property. Coherency property of Value-at-Risk occurs only for elliptical class of distributions.

Risk measure Conditional Value-at-Risk (CVaR) is based on generalization of VaR. This is conditional mathematical expectation:

$$\mathrm{CVaR}(R) = E(R|R \leq \mathrm{VaR}(R))$$

The essence of VaR and CVaR is illustrated by picture at Figure 2.

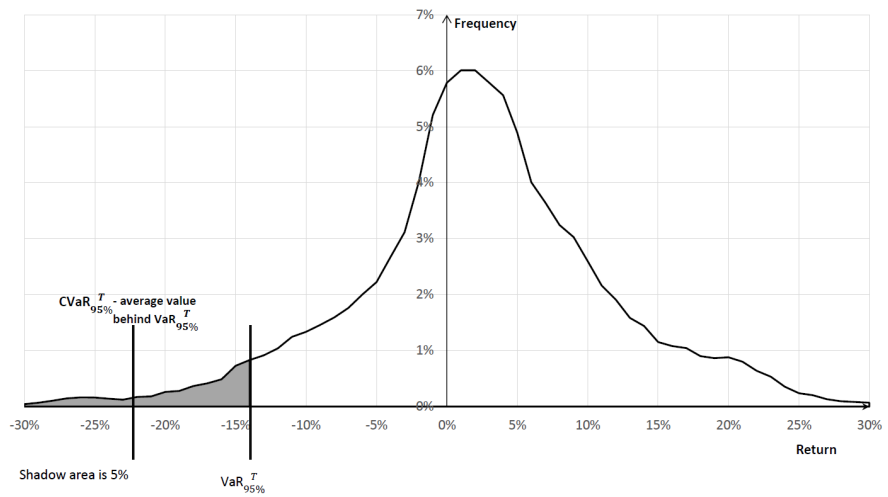

Figure 2. Essence of VaR and CVaR

Advantages of CVaR include coherency of this risk measure and more correct considering of possible losses.

Statistical estimations of VaR and CVaR for cryptocurrencies under consideration we present below in Table 3.

Table 3. Statistical estimations of VaR and CVaR (daily return, safety level – 95%; time period: 01.01.2018 – 17.08.2018)

Cryptocurrency	VaR	CVaR	Cryptocurrency	VaR	CVaR
BTC	-8.2%	-11.0%	ADA	-12.2%	-15.1%
ETH	-9.6%	-12.8%	XMR	-11.1%	-14.3%
XRP	-11.7%	-14.9%	ETC	-12.0%	-15.9%
BCH	-10.5%	-15.3%	TRX	-15.1%	-19.1%
EOS	-12.9%	-17.4%	MIOTA	-13.3%	-16.2%
XLM	-12.8%	-16.1%	DASH	-9.4%	-13.1%
LTC	-8.7%	-12.7%	NEO	-13.1%	-17.8%
USDT	-0.6%	-1.1%			

The ratio CVaR/VaR characterizes correspondence between "catastrophic" losses and maximal losses at the frameworks of 95% safety level. Our consideration shows that ratio belongs to interval [1.22; 1.81]. Such interval is relatively wide, so cryptocurrencies are quite different in behaviour of left pdf tails.

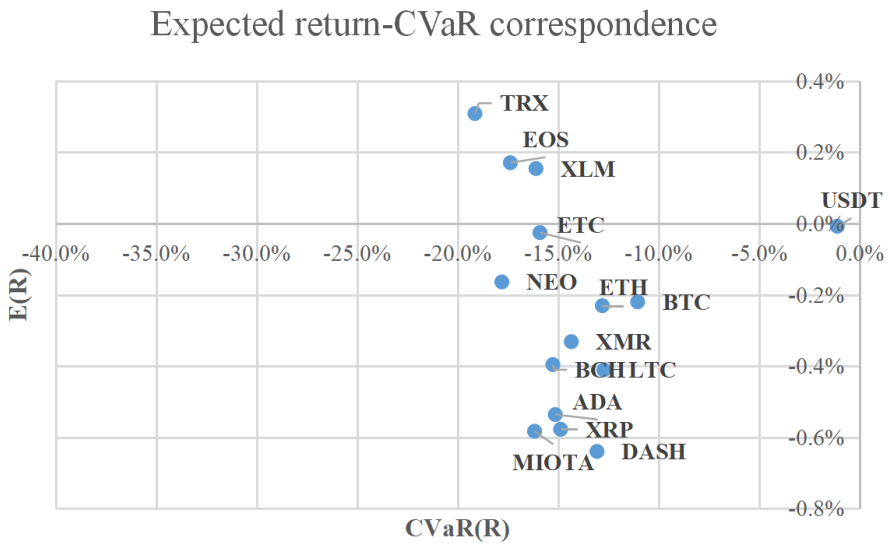

Figure 3. Correspondence "expected return – CVaR"

4 Risk measurement at the frameworks of sensitivity analysis

One of the most important approaches for investment risk measurement is based on sensitivity analysis. The importance of this approach is based on possibility to structure risk into systematic and nonsystematic risks. Systematic risk reflects impact of market changes to return of an investigated asset. Sensitivity analysis involves procedures for assessment of such impacts. Classical approach consists in using a linear regression model for return:

$$R_A = \alpha_A + \beta_A R_I + \varepsilon_A$$

where

- R_I indicates return of some market index (source of systematic risk);
- R_A is return of investment asset;
- β_A - coefficient of sensitivity (more precisely, this coefficient explains sensitivity numerically);
- α_A- coefficient of linear regression;
- ε_A is a random variable which indicate "own" – nonsystematic risk (not caused by the index).

One of the crucial suppositions in this model is independence between random variables R_I and ε_A. So, covariance between those random variables equals 0.

Risk structuring on systematic and nonsystematic risk can be obtained after applying operator of variance to formula for R_A:

$$\sigma^2(R_A) = \beta_A^2 \cdot \sigma^2(R_I) + \sigma^2(\varepsilon_A).$$

Ratios

$$\frac{\beta_A^2 \cdot \sigma^2(R_I)}{\beta_A^2 \cdot \sigma^2(R_I) + \sigma^2(\varepsilon_A)}$$

and

$$\frac{\sigma^2(\varepsilon_A)}{\beta_A^2 \cdot \sigma^2(R_I) + \sigma^2(\varepsilon_A)}$$

will be indicators of significance of systematic risk and nonsystematic risk correspondingly. Ratios are measured as percentages.

In our research we applied such approach to the index model which is based on the cryptocurrencies index CRIX (Trimborn and Härdle (2017)). The results – beta-coefficients to index CRIX are given in Table 4.

Table 4. Statistical estimation of beta-coefficient daily return, period: 01.01.2018–17.08.2018

Cryptocurrency	Beta coefficient	Cryptocurrency	Beta coefficient
BTC	0.1262	ADA	0.2382
ETH	0.1361	XMR	0.0544
XRP	0.1840	ETC	0.1214
BCH	0.1485	TRX	0.2518
EOS	0.1925	MIOTA	0.0495
XLM	0.1579	DASH	0.0753
LTC	0.0852	NEO	-0.0353
USDT	-0.0171		

Structure of the risk is presented at the Figure 4.

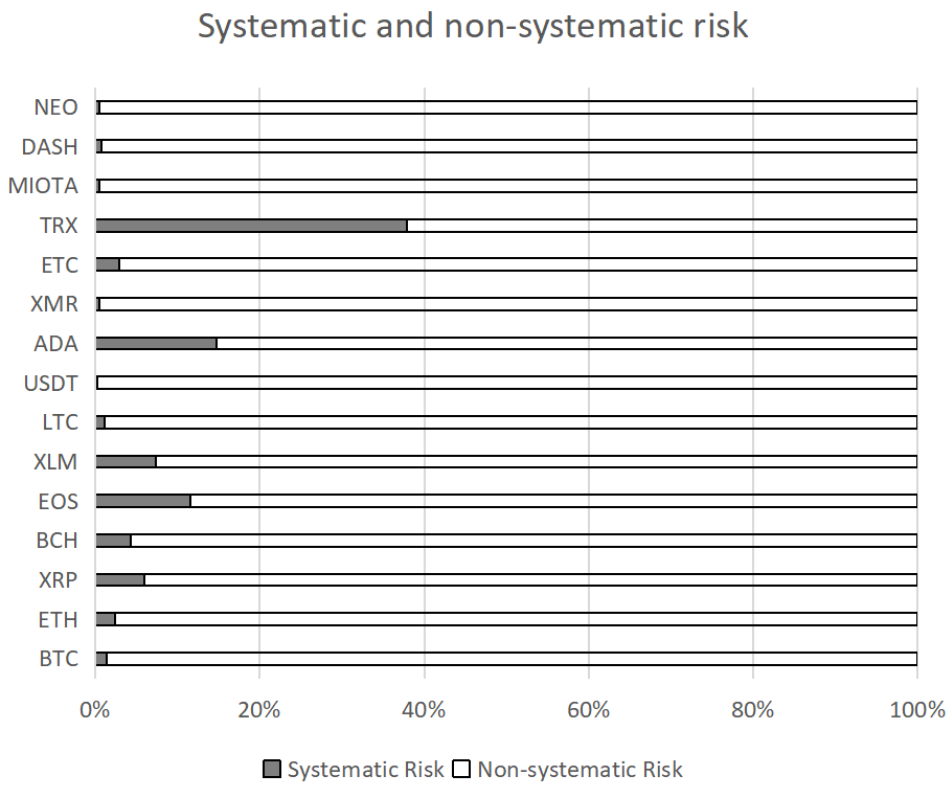

Figure 4. Correspondence between systematic and nonsystematic risks

Results show that nonsystematic risks are dominated.

5 Conclusions

The measurement of investment risk is multifaceted task which supposed to apply different approaches. Each approach points out specific features of risk.

The application of different approaches for the risk measurement of basic cryptocurrencies makes it possible to form some conclusions. First conclusion indicates a relatively high level of risk at the frameworks of volatility and significant outliers. Most cryptocurrencies demonstrate 5%–10% of standard deviation. The ratio of Range/Interquartile range is also relatively high. Kurtosis demonstrates high values. The risk measurements on the base of VaR and CVaR also indicate their values as high as ratio CVaR/VaR. All these results can be explained by significant outliers.

Second conclusion concerns exclusively high proportion of nonsystematic risk. Economically this can be explained by absence of meaningful factor which affects for all cryptocurrencies. This also revealed in low values of beta-coefficients in CRIX index model. On the other hand, such results may be raised from imperfection of index construction.

Third conclusion, maybe the most interesting, consists in fact that "classical" relationship between risk and return cannot be identified.

References

Artzner P., Delbaen F., Eber J., Heath D., (1999), *Coherent measures of risk,* Mathematical Finance, vol. 4, 203–228.

Chan S., Chu J., Nadarajah S., Osterrieder J., (2017), *A statistical analysis of cryptocurrencies*, Journal of Risk Financial Management, vol. 10, no. 12, 1–23.

Fischer T., (2003), *Risk capital allocation by coherent risk measures based on one-sided moments,* Insurance: Mathematics and Economics, vol. 32, 135–146.

Gangwal S., (2018), *Analyzing the effects of adding bitcoin to portfolio*, International Journal of Economics and Management Engineering, vol. 10, no. 10, 3519–3532.

Holton G., (2003), *Value-at-Risk: Theory and Practice,* Elsevier Science, 2003.

Kaminskyi A., (2006), *Financial Risk Modeling*, Kyiv National Taras Shevchenko University (in Ukrainian).

Lee D., Guo L., Wang Y., (2018), *Cryptocurrency: A new investment opportunity?*,Journal of Alternative Investments, 20(3), 16–40.

Markowitz H., (1959), *Portfolio Selection: Efficient Diversification of Investment*, John Wiley & Sons Ltd.

Rockafellar R., Uryasev S., (2000), *Optimization of conditional Value-At-Risk*, The Journal of Risk, vol. 2, no. 3, 21–41.

Scott R., Horvath P., (1980), *On the direction of preference for moments of higher order than the variance*, Journal of Finance, vol. 35, 915–919.

Szego G. (ed.), (2004), *Risk Measures for the 21st Century*, John Wiley & Sons Ltd.

Trimborn S., Härdle W., (2017), *CRIX an index for cryptocurrencies*, SFB 649 Economic Risk Discussion Paper.

Trimborn S., Mingyang L., Härdle W., (2017), *Investing with Cryptocurrencies – A Liquidity Constrained Investment Approach*, SFB 649 Economic Risk Discussion Paper.

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Entropy, Markov chains and Markov operators

Keywords: Boltzmann-Gibbs entropy, Shannon entropy, Markov chain, Markov operator, bivariate geometric distribution

Abstract

We discuss here the concept of entropy and its application to Markov chains and Markov operators. We focus our attention on the Boltzmann-Gibbs and the Shannon entropy. The Kolmogorov-Sinai and Makarov entropy will be discussed in a separate article.

1 Preliminaries

We begin with some formal stuff. So the first thing to declare is a measure space $(X,\mathscr{A},\mu)$ where X is a phase space, usually a subset of $\mathbb{R}^d$, $\mathscr{A}$ denotes the σ-algebra of subsets of X nad μ is a measure on $\mathscr{A}$ that is assumed to be σ-finite. If $f\in L^1(X,\mathscr{A},\mu)$ it means that $||f||:=\int_X|f(x)|dx<\infty$. A density is $f\geq 0$ with $||f||=1$. An observable $\mathscr{O}$ is a map from X to $\mathbb{R}$ (e.g. energy, pressure, temperature, etc). The Boltzmann-Gibbs entropy of the density f is defined as

$$H(f)=-\int_X f(x)\log f(x)dx. \tag{1}$$

Sometimes it is also termed "differential entropy" (usually in mathematical literature). It is important to note that it has not the same properties as the Shannon entropy, see definition (8) below. For example, it may be negative. If f is the density of the gaussian distribution $N(\mu,\sigma^2)$, then $H(f)=\log(\sigma\sqrt{2\pi e})$ and $H(f)<0$ for $\sigma<1/\sqrt{2\pi e}$, see also Fig. 1. Definition (1) can be formalized in the following way $H(f)=\int_X\eta(f(x))dx$, where

$$\eta(x)=\begin{cases}-x\log x, & x>0\\ 0, & x=0.\end{cases}$$

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Many properties of the entropy H is derived from the Gibbs inequality

$$w - w\log w \leq v - w\log v, \quad w,v > 0, \tag{2}$$

and the integrated Gibbs inequality

$$-\int_X f(x)\log f(x)dx \leq -\int_X f(x)\log g(x)dx, \tag{3}$$

where f,g are pdfs such that $\eta(f)$ and $\eta(g)$ are integrable. It is well known, see e.g. Mackey (2003), that if $\mu(X) < \infty$ then the uniform density $f_*(x) = 1/\mu(X)$, $x \in X$, maximizes the Boltzman-Gibbs entropy. This mean that if $f \neq f_\star$ then $H(f) < H(f_*)$.

Example 1 *By $Geo(p)$ we denote the geometric distribution with parameter p. So if $X \sim Geo(p)$, then $\mathbb{P}(X = k) = p(1-p)^{k-1}$, $k = 1,2,\ldots$ and $p \in (0,1)$. We use here rather the definition of the Shannon entropy (8) than (1), but it makes no difference in this case. Also, the notation $H(p)$ is more convenient than $H(f)$ or $H(X)$. Thus*

$$H(p) = -\sum_{k=1}^{+\infty} p(1-p)^{k-1}\log[p(1-p)^{k-1}] = -\log(p)\sum_{k=1}^{+\infty} p(1-p)^{k-1}$$
$$-p\log(1-p)\sum_{k=1}^{+\infty}(k-1)(1-p)^{k-1},$$

and using $\sum_{k=1}^{\infty} kx^k = x/(1-x^2)$, for $|x| < 1$, we get

$$H(p) = -\frac{(1-p)}{p}\log(1-p) - \log p, \quad p \in (0,1). \tag{4}$$

From this $\lim_{p\to 0+} H(p) = +\infty$ and $\lim_{p\to 1} H(p) = 0$, see Fig. 1.

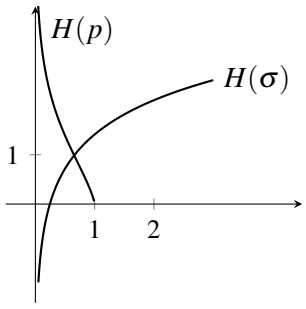

Figure 1. The Shannon entropy $H(p)$ (nonnegative) of geometric distribution and the Boltzman-Gibbs entropy of normal distribution $N(\mu,\sigma^2)$. Note that $H(\sigma) < 0$ for $\sigma < 1/\sqrt{2\pi e}$

We have the following theorem, see Theorem 2.2 in Mackey (2003). It explains how to find the probability distribution with maximal entropy under certain constraints.

Theorem 2 *Assume that $\mathscr{O}(x) \geq 0$ is a measurable function and that*

$$< \mathscr{O} >= \int_X \mathscr{O}(x) f(x) dx. \tag{5}$$

Then the maximum of the entropy $H(f)$, subject to the constraint (5), occurs for the density

$$f_*(x) = Z^{-1} e^{-\nu \mathscr{O}(x)} \quad with \quad Z = \int_X e^{-\nu \mathscr{O}(x)} dx,$$

where ν is implicitly determined from

$$< \mathscr{O} >= Z^{-1} \int_X \mathscr{O}(x) e^{-\nu \mathscr{O}(x)} dx.$$

Additionally, $H(f_*) = \log Z + \nu < \mathscr{O} >$.

Remark 3 *It should be noted that for any discrete random variable X with values in $\{1,2,3,\ldots\}$ we have $\mathbb{E}(X) \geq 1$. Namely,*

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} n p_n \geq \sum_{n=1}^{\infty} p_n = 1.$$

Example 4 *Let $X = \{1,2,3,\ldots\}$ and $\mathscr{O}(n) = n$, then $< \mathscr{O} >= \mu$. It is assumed that $\mu \geq 1$. We have*

$$f_*(n) = \frac{1}{Z} e^{-\nu n}, \quad Z = \sum_{n=1}^{+\infty} e^{-\nu n} = \frac{1}{e^{\nu} - 1},$$

where ν is the solution to

$$\mu = (e^{\nu} - 1) \sum_{n=1}^{+\infty} n e^{-\nu n} = \frac{(e^{\nu} - 1) e^{-\nu}}{(1 - e^{-\nu})^2} = \frac{1}{1 - e^{-\nu}}.$$

From this we get $e^{\nu} = \mu/(\mu - 1)$, $Z = \mu - 1$ and finally

$$f_*(n) = \frac{1}{\mu} \left(1 - \frac{1}{\mu}\right)^{n-1}, \quad n = 1,2,\ldots$$

This means that among all probability distributions on $\mathbb{N}$ with a given mean μ the geometric distribution has the biggest entropy. □

If a vector (X,Y) has density $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, meaning that X and Y are independent, then $H(f_{X,Y}) = H(f_X) + H(f_Y)$. Indeed, by definition

$$\begin{aligned} H(f_{X,Y}) &= -\int\int f_X(x)f_Y(y)\log\left(f_X(x)f_Y(y)\right)dxdy \\ &= -\int\int f_X(x)f_Y(y)\log f_X(x)dxdy - \int\int f_X(x)f_Y(y)\log f_Y(y)dxdy \\ &= H(f_X) + H(f_Y). \end{aligned}$$

For the normal vector $N(\vec{\mu},\Sigma)$ the entropy equals $\frac{1}{2}\log[\det(2\pi e\Sigma)]$, where Σ is a covariance matrix of $\vec{X}$. Note that this does not depend on $\vec{\mu}$. In particular, for (X,Y) with

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

we have $H(f_{X,Y}) = 1 + \log(2\pi\sigma_1\sigma_2) + \frac{1}{2}\log(1-\rho^2)$. Since $X \sim N(\mu_1,\sigma_1^2)$ we have $H(f_X) = \frac{1}{2}(1+\log(2\pi\sigma_1))$ and similarly $H(f_Y) = \frac{1}{2}(1+\log(2\pi\sigma_2))$. Hence, in this case and Boltzmann-Gibbs entropy

$$H(f_X) + H(f_Y) - H(f_{X,Y}) = \tfrac{1}{2}\ln(1-\rho^2) \leq 0.$$

If $X \sim Geo(p_1)$ and $Y \sim Geo(p_2)$ and X,Y are independent, then

$$\mathbb{P}(X=m, Y=n) = p_1(1-p_1)^{m-1}p_2(1-p_2)^{n-1}, \quad m,n \in \mathbb{N},$$

and $H(X,Y) = H(X) + H(Y)$. In other words: $H(X,Y) = H(p_1) + H(p_2)$ by (4). Now, let (X,Y) be a random vector with a probability mass function

$$P(X=m, Y=n) = p_{m,n}, \quad m,n \in \mathbb{N},$$

and given $\mu_1 = \mathbb{E}(X)$, $\mu_2 = \mathbb{E}(Y)$. Then it turns out that (X,Y) with

$$p_{m,n} = \frac{1}{\mu_1\mu_2}\left(1-\frac{1}{\mu_1}\right)^{m-1}\left(1-\frac{1}{\mu_2}\right)^{n-1}, \quad m,n \in \mathbb{N}$$

has the biggest entropy. This is because of the following theorem, see Theorem 2.3 in Mackey (2003). In fact it is a generalization of Theorem 2.

Theorem 5 *Suppose that $\mathscr{O}_i(x) \geq 0$, $i = 1,\ldots,m$ is a sequence of measurable functions with given averages $< \mathscr{O}_i(x) >$. Then the maximum entropy $H(f)$ for all densities f, subject to the conditions*

$$< \mathscr{O}_i > = \int_X \mathscr{O}_i(x)f(x)dx, \tag{6}$$

occurs for the density

$$f_*(x) = Z^{-1}\prod_{i=1}^{m} e^{-\nu_i \mathscr{O}_i(x)} \quad with \quad Z = \int_X \prod_{i=1}^{m} e^{-\nu_i \mathscr{O}_i(x)} dx,$$

where ν_i'*s are determined from*

$$< \mathscr{O}_i >= Z^{-1}\int_X \mathscr{O}_i(x) e^{-\nu_i \mathscr{O}_i(x)} dx, \quad i = 1,\ldots,m.$$

In the following example we consider a situation when both X and Y have geometric distribution and are positively correlated. In addition, we calculate the entropy of (X,Y).

Example 6 *(Basu-Dhar bivariate geometric distribution) Consider a random vector* (X,Y)*, supported on* $\mathbb{N}^2$*, which has the following distribution*

$$p_{m,n} = \begin{cases} p_1^{m-1}(p_2p_{12})^{n-1}q_1(1-p_2p_{12}), & m < n, \\ (p_1p_2p_{12})^{m-1}(1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}), & m = n, \\ p_2^{n-1}(p_1p_{12})^{m-1}q_2(1-p_1p_{12}), & m > n, \end{cases} \tag{7}$$

where $p_{m,n} = \mathbb{P}(X = m, Y = n)$ *and* $p_1, p_2, p_{12} \in (0,1)$ *are independent parameters with* $q_1 = 1-p_1$*,* $q_2 = 1-p_2$*. It was introduced in Basu and Dhar (1995) and is called the bivariate geometric distribution, due to the fact that X and Y have geometric distribution (we will see it in a while). The joint survival function of* (X,Y) *is*

$$\mathbb{P}(X > m, Y > n) = p_1^m p_2^n p_{12}^{\max(m,n)}.$$

Incidently, one interesting feature of this distribution is the loss memory property. Namely, for any $k,m,n \in \mathbb{N}$*, we have*

$$\begin{aligned} \mathbb{P}(X > m+k, Y > n+k | X > m, Y > n) &= \frac{\mathbb{P}(X > m+k, Y > n+k)}{\mathbb{P}(X > m, Y > n)} \\ &= \frac{p_1^{m+k}p_2^{n+k}p_{12}^{\max(m+k,n+k)}}{p_1^m p_2^n p_{12}^{\max(m,n)}} \\ &= (p_1p_2p_{12})^k \end{aligned}$$

which is $\mathbb{P}(X > k, Y > k)$*. Before calculating the entropy of* (X,Y) *we will find the distribution of X. We have*

$$\begin{aligned} \mathbb{P}(X = m) = \sum_{n=1}^{+\infty} p_{m,n} &= (p_1p_{12})^{m-1}(1-p_1p_{12})(1-p_2^{m-1}) + p_1^{m-1}q_1(p_2p_{12})^m \\ &\quad + (p_1p_2p_{12})^{m-1}(1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}) \\ &= (1-p_1p_{12})(p_1p_{12})^{m-1}, \quad m = 1,2,3,\ldots \end{aligned}$$

therefore $X \sim Geo(1-p_1p_{12})$. *Similarly* $Y \sim Geo(1-p_2p_{12})$. *We also have*

$$\mathbb{E}(XY) = \frac{1-p_1p_2p_{12}^2}{(1-p_1p_{12})(1-p_2p_{12})(1-p_1p_2p_{12})},$$

and in consequence

$$cov(X,Y) = \frac{p_1p_2p_{12}(1-p_{12})}{(1-p_1p_{12})(1-p_2p_{12})(1-p_1p_2p_{12})} \geq 0.$$

Denote $\mathbb{N}^2 = D_1 + D_2 + D_3$, *where* $D_1 = \{(m,n) \in \mathbb{N}^2 : m > n\}$, $D_2 = \{(m,n) \in \mathbb{N}^2 : m = n\}$ *and* $D_3 = \{(m,n) \in \mathbb{N}^2 : m < n\}$. *We have, see also* (9),

$$H(X,Y) = \underbrace{-\sum_{(m,n)\in D_1} p_{m,n}\log p_{m,n}}_{=H_{D_1}} \underbrace{-\sum_{(m,n)\in D_2} p_{m,n}\log p_{m,n}}_{=H_{D_2}} \underbrace{-\sum_{(m,n)\in D_3} p_{m,n}\log p_{m,n}}_{=H_{D_3}}.$$

We begin with H_{D_2}. *By* (7)

$$H_{D_2} = -c\sum_{m=1}^{\infty}(p_1p_2p_{12})^{m-1}\log\left[c\cdot(p_1p_2p_{12})^{m-1}\right],$$

where $c := 1-p_1p_{12}-p_2p_{12}+p_1p_2p_{12}$. *Direct computation gives*

$$H_{D_2} = \frac{-c}{(1-p_1p_2p_{12})^2}\left[\log c + p_1p_2p_{12}\log\left(\frac{p_1p_2p_{12}}{c}\right)\right].$$

As for D_1, *with* $b := q_2(1-p_1p_{12})$, *we have*

$$H_{D_1} = \underbrace{-b\log b\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}p_2^{n-1}(p_1p_{12})^{m-1}}_{=S_1} \underbrace{-b\log p_2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}(n-1)p_2^{n-1}(p_1p_{12})^{m-1}}_{=S_2} \underbrace{-b\log(p_1p_{12})\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}(m-1)p_2^{n-1}(p_1p_{12})^{m-1}}_{=S_3}.$$

Performing calculations we obtain

$$S_1 = -\frac{p_1 p_{12} q_2}{(1-p_1 p_2 p_{12})} \log[q_2(1-p_1 p_{12})], \quad S_2 = -\frac{p_2 q_2 (p_1 p_{12})^2 \log p_2}{(1-p_1 p_2 p_{12})^2}$$

and

$$S_3 = -\frac{p_1 p_{12} q_2 \left(1-p_2(p_1 p_{12})^2\right) \log(p_1 p_{12})}{(1-p_1 p_{12})(1-p_1 p_2 p_{12})^2}.$$

In a similar way we find H_{D_3}. Finally we have

$$\begin{aligned} H(X,Y) = &- \frac{p_{12}\left[p_1 q_2 \log[q_2(1-p_1 p_{12})] + p_2 q_1 \log[q_1(1-p_2 p_{12})]\right]}{(1-p_1 p_2 p_{12})} \\ &- \frac{p_{12}^2 \left(p_1^2 p_2 q_2 \log p_2 + p_1 p_2^2 q_1 \log p_1\right)}{(1-p_1 p_2 p_{12})^2} \\ &- \frac{p_1 q_2 p_{12}(1-p_2(p_1 p_{12})^2)\log(p_1 p_{12})}{(1-p_1 p_{12})(1-p_1 p_2 p_{12})^2} \\ &- \frac{p_2 q_1 p_{12}(1-p_1(p_2 p_{12})^2)\log(p_2 p_{12})}{(1-p_2 p_{12})(1-p_1 p_2 p_{12})^2} \\ &- \frac{c}{(1-p_1 p_2 p_{12})^2}\left[\log c + p_1 p_2 p_{12} \log\left(\frac{p_1 p_2 p_{12}}{c}\right)\right], \end{aligned}$$

where $c = 1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12}$. □

2 Entropy and Markov chains

The Shannon entropy of a discrete random variable X with values in $\{x_1, \ldots, x_n\}$ and a probability distribution $\vec{p} = (p_1, \ldots, p_n)$, i.e. $\mathbb{P}(X = x_i) = p_i$, $i = 1, \ldots, n$, is defined as

$$H(X) := -\sum_{i=1}^{n} p_i \log p_i \tag{8}$$

with the notion that $0 \cdot \log 0 = 0$ if $p_i = 0$ for some i. We also use $H(\vec{p})$ for $H(X)$. Directly from definition we have that $0 \leq H(X) \leq \log(n)$. Note that $H(X) = 0$ if and only if $X = const$ and $H(X) = \log(n)$ only if $\mathbb{P}(X = x_i) = 1/i$ for $i = 1, \ldots, n$. The definition of (8) extends also to infinite sum. However note that then the Shannon entropy may be infinite. Namely, define $C_p := \sum_{n=2}^{\infty} (n \log^p n)^{-1}$, for $p \in \mathbb{R}$. It is know that $C_p < \infty$ for $p > 1$ and $C_p = \infty$ if $p \leq 1$. Let X be a rv with $\mathbb{P}(X = n) = (C_p n \log^p n)^{-1}$, where $n = 2, 3, \ldots$ and $p > 1$. We have

$$\begin{aligned} H(X) &= -\sum_{n=2}^{\infty} \frac{1}{C_p n (\log n)^p} \log\left(\frac{1}{C_p n (\log n)^p}\right) \\ &= \frac{1}{C_p} \sum_{n=2}^{\infty} \left(\frac{\log(C_p)}{n(\log n)^p} + \frac{1}{n(\log n)^{p-1}} + \frac{p \log(\log n)}{n(\log n)^p}\right) = \infty \end{aligned}$$

for $p \in (1,2]$. Analogously to (8), the entropy of a random vector $(X_1,\ldots,X_n)$ is

$$H(X_1,\ldots,X_n) := -\sum_{i_1,\ldots,i_n} p_{i_1,\ldots,i_n} \log p_{i_1,\ldots,i_n}, \tag{9}$$

where $p_{i_1,\ldots,i_n} = \mathbb{P}(X_1 = x_{i_1},\ldots,X_n = x_{i_n})$. If $\mathscr{X} = \{X_n\}_{n\geq 1}$ is a sequence if random variables, then the entropy rate of $\mathscr{X}$ is defined as

$$H(\mathscr{X}) := \lim_{n\to+\infty} \frac{1}{n} H(X_1,\ldots,X_n) \tag{10}$$

if the above limit exists. For X,Y we define $H(X|Y) := H(X,Y) - H(Y)$ and

$$H'(\mathscr{X}) := \lim_{n\to+\infty} H(X_n|X_1,\ldots,X_{n-1}), \tag{11}$$

if this limit exists. We recall that a sequence of random variables $X_1, X_2, \ldots$ is stationary if for all n, the distribution of the vector $(X_1,\ldots,X_n)$ is the same as that of $(X_{1+k},\ldots,X_{n+k})$ for every $k \geq 1$. For stationary sequences we have the theorem, see e.g. Theorem 12.1.5 in Bremaud (2017)

Theorem 7 *If $\mathscr{X} = \{X_n\}_{n\geq 1}$ is a stationary sequence of random variables, then both* (10), (11) *exist and $H(\mathscr{X}) = H'(\mathscr{X})$.*

We want to state the analogous theorem for Markov chains, however we need just few definitions. Assume that each $X_n \in \mathscr{R}$. We say that $X_1, X_2, \ldots$ is a time invariant Markov chain if

$$\mathbb{P}(X_{n+1} = x_{n+1}|X_n = x_n,\ldots,X_1 = x_1) = \mathbb{P}(X_{n+1} = x_{n+1}|X_n = x_n)$$

for each $n \in \mathbb{N}$, all $x_1,\ldots,x_{n+1} \in \mathscr{R}$ and

$$\mathbb{P}(X_2 = y|X_1 = x) = \mathbb{P}(X_3 = y|X_2 = x) = \mathbb{P}(X_4 = y|X_3 = x) = \ldots,$$

for all $x,y \in \mathscr{R}$. Thus a time invariant Markov chain is characterized by its initial state and a probability transition matrix $P = (p_{ij})$, where

$$p_{ij} = \mathbb{P}(X_n = x_j|X_{n-1} = x_i), \quad i,j \in I.$$

We say that a probability distribution $\pi = (\pi_1, \pi_2, \ldots)$ is the stationary distribution of a Markov chain P if

$$\pi = \pi P \quad \Leftrightarrow \quad \pi_j = \sum_{i\in I} \pi_i p_{ij}, \quad \forall j \in I.$$

For a stationary Markov chain the following theorem holds, see e.g. Theorem 4.2.4 in Cover and Thomas (2006).

Theorem 8 *Suppose that a stationary Markov chain $\mathscr{X} = \{X_n\}_{n\geq 1}$, with transition matrix P, has a stationary distribution π. Then*

$$H(\mathscr{X}) = -\sum_{i,j} \pi_i p_{ij} \log p_{ij}. \tag{12}$$

We begin examples with a random walk on a graph without weights. For general case see Chapter 4 in Cover and Thomas (2006).

Example 9 *Consider a graph G and a particle that walks randomly from vertex to vertex of G. The random walk $\mathscr{X} = \{X_n\}_{n\geq 1}$, where $X_n \in \{v_1, \ldots, v_5\}$ (or we can simply put the numbers of vertices $\{1, \ldots, 5\}$) is a sequence of vertices of G. If $X_n = i$, the next vertex j is chosen from among the vertices connected to i with a probability proportional to the number of the edges connecting i to j, see Fig. 2.*

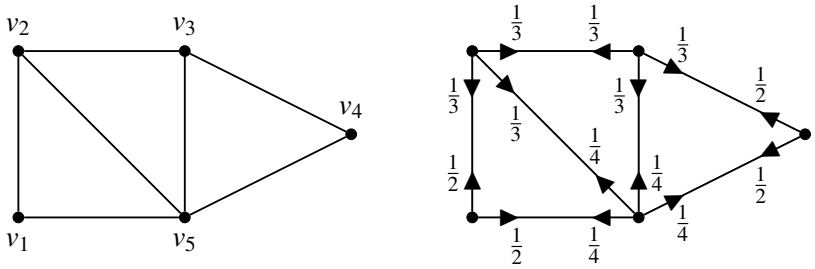

Figure 2. Random walk on a graph

This random walk is a Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

The stationary distribution of P is given by $\pi = (\frac{2}{14}, \frac{3}{14}, \frac{3}{14}, \frac{2}{14}, \frac{4}{14})$. By (12) we have

$$H(\mathscr{X}) = \frac{4}{14}\log 2 + \frac{6}{14}\log 3 + \frac{4}{14}\log 4 \approx 1.064. \quad \square$$

Example 10 *Suppose that a particle jumps from 0 to n, $n \geq 1$ with probability p_n and then goes back to 0 in n steps as shown in Fig. 3. When a particle is at 0, then it again jumps to a natural number and goes back to 0 and repeats this scheme infinitely many times. This process is a Markov chain with values in $\{0, 1, 2, \ldots\}$*

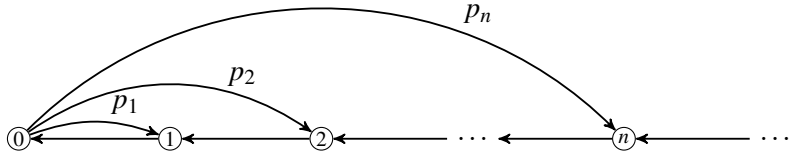

Figure 3. Markov chain on {0,1,2,...}. An illustration of Example 10

and infinite transition matrix P

$$P = \begin{bmatrix} 0 & p_1 & p_2 & p_3 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where $\vec{p} = (p_1, p_2, \dots)$ *is a probability distribution, i.e.* $\sum_{i=1}^{\infty} p_i = 1$ *and each* $p_i \geq 0$. *We will find the stationary distribution of P. Denote* $\pi = (\pi_0, \pi_1, \pi_2, \dots)$. *Solving* $\pi = \pi P$ *we get*

$$\pi P = (\pi_1, \pi_0 p_1 + \pi_2, \pi_0 p_2 + \pi_3, \dots) \Rightarrow (\pi P)_n = \begin{cases} \pi_1, & n = 0, \\ \pi_0 p_n + \pi_{n+1}, & n \geq 1. \end{cases}$$

From this $\pi_1 = \pi_0$, $\pi_2 = \pi_0(1 - p_1)$ *and so on. Hence* π_0 *is arbitrary,* $\pi_1 = \pi_0$ *and*

$$\pi_n = \pi_0 \left(1 - \sum_{i=1}^{n-1} p_i \right), \quad n \geq 2.$$

Now we check the condition $\sum_{n=0}^{\infty} \pi_n = 1$. *If* $\mu := \sum_{n=1}^{\infty} n p_n < \infty$ *then*

$$\sum_{n=0}^{\infty} \pi_n = \pi_0(\mu - 1),$$

and $\pi_0 = 1/(\mu - 1)$. *Thus the stationary distribution of P is given by*

$$\pi_0 = \pi_1 = \frac{1}{\mu - 1}, \qquad \pi_n = \frac{1}{\mu - 1} \left(1 - \sum_{i=1}^{n-1} p_i \right), \quad n \geq 2. \tag{13}$$

In consequence

$$H(\mathscr{X}) = -\sum_{j=1}^{\infty} \pi_0 p_{0,j} \log p_{0,j} = -\pi_0 \sum_{j=1}^{\infty} p_i \log p_i = \frac{H(\vec{p})}{\mu - 1}.$$

In particular, for a geometric distribution Geo(p) *we have*

$$H(\mathscr{X}) = -\frac{p}{(1-p)} \log p - \log(1-p), \quad p \in (0,1). \quad \square$$

3 Conditional entropy and Markov operators

The conditional entropy of a density f with respect to a density g is defined as follows

$$H_c(f|g) = -\int_X f(x)\log\left[\frac{f(x)}{g(x)}\right]dx. \tag{14}$$

The first important property of H_c is that $H_c(f|g) \leq 0$, for any densities f, g (by the Gibbs inequality (3)). Additionally $H_c(f|g) = 0$ only if $f = g$. The second property is that if $g = 1/\mu(X)$ then $H_c(f|g) = H(f) - \log\mu(X)$. Thus, when $\mu(X) = 1$, then $H_c(f|1) = H(f)$, see Mackey (2003).

A linear operator $P : L^1(X,\mathscr{A},\mu) \to L^1(X,\mathscr{A},\mu)$ is called a Markov operator if it satisfies

(1) if $f \geq 0$, then $Pf \geq 0$;

(2) if $f \geq 0$, then $||Pf|| = ||f||$.

Consider l^1 space. Recall that it is a Banach space that consists of infinite sequences $x = (x_1, x_2, \ldots)$ such that

$$||x|| := \sum_{i=1}^{\infty} |x_i| < \infty.$$

We write $x \geq 0$ if $x_n \geq 0$ for each $n = 1, 2, \ldots$. An $x \in l^1$ is termed a density if $x \geq 0$ and $||x|| = 1$. For more information about Markov operators in l^1 see Rudnicki (2014).

Example 11 *Suppose that $P = (p_{ij})$ is a infinite transition matrix, i.e. its entries are non-negative and*

$$\sum_{j=1}^{\infty} p_{ij} = 1, \quad i = 1, 2, \ldots. \tag{15}$$

We define an operator P on l^1 by

$$Px := x \cdot P = \left(\sum_{i=1}^{\infty} x_i p_{i1}, \sum_{i=1}^{\infty} x_i p_{i2}, \ldots\right)$$

It is a Markov operator. If $x \geq 0$ and $x \in l^1$, then by non-negativity and (15) *we have*

$$||Px|| = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_i p_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i p_{ij} = \sum_{i=1}^{\infty} x_i = ||x||. \quad \square$$

We have the following theorem, see Theorem 3.1 in Mackey (2003). In fact, this is a theorem by J. Voigt, see Voigt (1981).

Theorem 12 *If P is a Markov operator, then*

$$H_c(Pf|Pg) \geq H_c(f|g) \tag{16}$$

for $f \geq 0$ and all densities g.

Example 13 *Consider the transition matrix*

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

It is also a Markov operator as explained in Example 11. If $x \in l^1$, then

$$(Px)_n = \begin{cases} \frac{1}{2}x_1, & n = 1, \\ \frac{1}{2}(x_{n-1} + x_n), & n \geq 2. \end{cases} \tag{17}$$

Take for example $x_n = (\frac{1}{2})^n$ and $y_n = 2(\frac{1}{3})^n$, where $n \geq 1$. It is clear that $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ are densities. Thus

$$\begin{aligned} H_c(x|y) &= -\sum_{n=1}^{\infty} x_n \log \frac{x_n}{y_n} = -\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \log\left[\frac{1}{2}\left(\frac{3}{2}\right)^n\right] \\ &= \log \tfrac{8}{9} = 3\log 2 - 2\log 3 \approx -0.1177. \end{aligned}$$

From (17) *we have*

$$(Px)_n = \begin{cases} \frac{1}{4}, & n = 1, \\ \frac{3}{2}(\frac{1}{2})^n, & n \geq 2, \end{cases} \qquad (Py)_n = \begin{cases} \frac{1}{3}, & n = 1, \\ 4(\frac{1}{3})^n, & n \geq 2. \end{cases}$$

Therefore

$$\begin{aligned} H_c(Px|Py) &= -\frac{1}{4}\log\frac{3}{4} - \frac{3}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \log\left[\frac{3}{8}\left(\frac{3}{2}\right)^n\right] \\ &= 5\log 2 - \tfrac{13}{4}\log 3 \approx -0.1047, \end{aligned}$$

and finally

$$H_c(Px|Py) - H_c(x|y) = 2\log 2 - \tfrac{5}{4}\log 3 = \log \tfrac{4}{\sqrt[4]{243}} > 0. \quad \square$$

4 Appendix 1

As for the Shannon entropy, see (8) and (9) for definition, we have the following properties (see Cover and Thomas (2006) or Bremaud (2017)):

(1) if $X_1, X_2, \ldots, X_n$ are independent, then

$$H(X_1, \ldots, X_n) = H(X_1) + \ldots + H(X_n).$$

(2) in general however we have (no independence assumed)

$$H(X_1, \ldots, X_n) \leq H(X_1) + \ldots + H(X_n).$$

Furthermore, the equality holds only if $X_1, \ldots, X_n$ are independent.

Define $H(X|Y) = H(X,Y) - H(Y)$. From this $H(X|Y) \leq H(X)$ and

(3) for rv's X, Y

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

(4) for $X_1, \ldots, X_{n+1}$, the following holds

$$H(X_1, \ldots, X_{n+1}) = H(X_1, \ldots, X_n) + H(X_{n+1}|X_1, \ldots, X_n)$$

(5) if $n \geq 2$, then

$$H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_1, \ldots, X_{n-1}).$$

5 Appendix 2

We solve here the problem 1.9.2 from Norris (1997).
Problem 1.9.2 Two particles X and Y perform independent random walks on the graph shown in the Fig. 4. So, for example, a particle at A jumps to B, C or D with equal probability $1/3$.

Find the probability that X and Y ever meet at a vertex in the following cases:

(a) X starts at A and Y starts at B;

(b) X starts at A and Y starts at E.

For $I = B, D$ let M_I denote the expected time, when both X and Y start at I, until they are once again both at I. Show that $9M_D = 16M_B$.
Solution. If a particle X performs a random walk on a graph with 21 vertices, then its state space is $I = \{A, B, C, \ldots\}$. Call this random walk a Markov chain on G

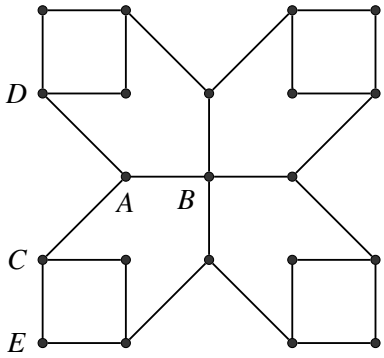

Figure 4. Two particles perform a random walk on G

or simply P. It is clear that P is irreducible. It is also periodic with period $d = 2$. Hence we can write $I = C_0 \cup C_1$, where $C_0 = \{A, E, \ldots\}$ and $C_1 = \{B, C, D, \ldots\}$, see Fig. 5 below. We have $|C_0| = 12$, $|C_1| = 9$. For example, if a particle starts from A it can return to A only in an even number of steps. In terms of classes: if a particle starts from class C_0 it can return to C_0 only in an even number of steps. The same is true for class C_1: starting from C_1, a particle can return to C_1 only in an even number of steps. However if a particle starts from C_0 it can reach C_1 only in an odd number of steps. Now we use the coupling method. Namely, if X_n and

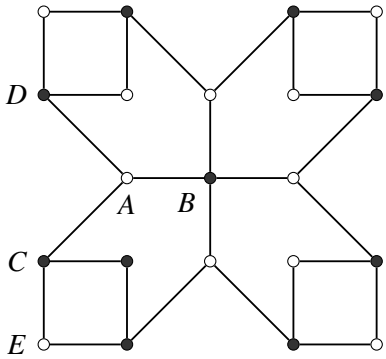

Figure 5. We have two classes: $C_0 = \{A, E, \ldots\}$**,** $C_1 = \{B, C, D, \ldots\}$

Y_n are independent Markov chains on I, then (X_n, Y_n) is a Markov chain on $I \times I$.

(a) Since A and B are in different classes, we have

$$\mathbb{P}\{\exists\, n \geq 1, \exists\, i \in I : (X_n, Y_n) = (i, i) | X_0 = A, Y_0 = B\} = 0.$$

This means that X and Y will never meet.

(b) Since $A, E \in C_0$ then the Markov chain (X_n, Y_n) with $(X_0, Y_0) = (A, E)$, considered only on $C_0 \times C_0$ is irreducible, recurrent and aperiodic. This is because

every finite closed class is recurrent, see Theorem 1.5.6 in Norris (1997). In such a Markov chain we have

$$\mathbb{P}\{\exists\, n \geq 1, \exists\, i \in I : (X_n, Y_n) = (i,i) | X_0 = A, Y_0 = E\} = 1$$

by Theorem 1.5.7 in Norris (1997). In conclusion: particle X starting at A and Y starting at E will meet at a vertex with probability 1.

Next, recall that the valency v_i of vertex $i \in I$ is the number of edges at i and $\sigma_G = \sum_{i\in I} v_i$ is the total valency of G. Thus we have $\sigma_G = 56$, $v_B = 4$ and $v_D = 3$.

Denote by m_B, m_D the expected return time to B and D respectively. Then, by Theorem 1.7.7 in Norris (1997) we have $m_B = 56/4 = 16$ and $m_D = 56/3 = 18\frac{2}{3}$.

Again, consider a Markov chain on $G \times G$. The number of vertices of the graph $G \times G$ is 21^2. We have

$$\sigma_{G\times G} = 56^2 = 3136, \quad v_{(B,B)} = 16, \quad v_{(D,D)} = 9$$

hence

$$M_B = \frac{3136}{16} = 256, \quad M_D = \frac{3136}{9} \quad \Rightarrow \quad 9M_D = 16M_B.$$

References

Basu A. P., Dhar S. K., (1995), *Bivariate geometric distribution*, Journal of Applied Statistical Science, 2, 33–44.

Bremaud P., (2017), *Discrete Probability Models and Methods*, Springer.

Cover T., Thomas J., (2006), *Elements of Information Theory*, Wiley.

Johnson N., Kotz S., Balakrishnan N., *Discrete Multivariate Distributions*, Wiley.

Mackey M. C., (2003), *Time's Arrow: The Origins of Thermodynamic Behavior*, Dover.

Norris J.R., (1997), *Markov chains*, Cambridge University Press.

Rudnicki R., (2014), *Modele i metody biologii matematycznej*, IM PAN, Warszawa.

Voigt J., (1981), *Stochastic operators, information and entropy*, Commun. Math. Phys., 81, 31–38.

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