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Introduction to Theory of Elasticity and Plasticity

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Chapter 1

Introduction

In this text, solid bodies are assumed continuous and the molecular structure of materials is neglected. We assume that the matter of a body can be indefinitely divisible and accept the idea of an infinitesimal volume of the body as well as the notion of a particle so that the we can make use of the mathematical calculus.

Under the mechanical (applied surface traction, body forces) and thermal (heating, cooling) interactions, a body deforms. Internal forces will be produced between the parts of this body. The intensity of the internal forces will be called *stress*. The amount of deformation that a body undergoes is described by *strain*. When the stresses are small and removed, the body will revert to its original shape. This behaviour is called *elasticity*.

A larger stress may cause *plastic* deformation. After a body undergoes plastic deformation, it will not revert to its original shape when the stress is removed. This phenomena is called *plasticity*.

Theory of Elasticity and Plasticity tries to explain the mechanical and geometrical changes of the body under interactions. Since deformable solids are special cases of continuous media, and since this is the first time the students of our faculty have deal with the subject, we will present in details the governing equations for the study of deformation and stress of a continuous material.

In Chapter 2, we will give a brief introduction to tensor calculus because tensors (and especially tensors of second order [9]) are constantly used in mechanics of continuous media. We are already familiar with the notion of scalars and vectors. A scalar is a quantity with magnitude only. Examples of scalars are temperature, time, mass...They are completely defined by only one value, e.g. degrees, seconds, kilograms...A free vector is a quantity with magnitude and direction. Examples of vectors are velocity, force, acceleration...they can be defined in a system of coordinate by three values, for example, three components on the axes which together specify both magnitude and directions. Tensors of second order, like strain, stress...are not familiar [7]. Stress which will be discussed in this course is not encountered in everyday life. It has nine components, six of which are independent and their values depend on the considered point and orientation relative to a set of reference axes. At a particular orientation, six components become zero and stress has only three principal components. These factors make stress difficult to understand without a deep consideration. Chapter 2 shows that scalars, vectors and tensors of second order belong to the same family of quantities: the tensors.

Kinematics is presented in Chapter 3. It is a study of the geometric changes or deformation in a body, without the consideration of forces causing the deformation. This is only a geometrical problem, no physical principle is involved. We'll start with the notion of motion, displacement then a measure of deformation: a strain tensor. We will study the strain-displacement relations and relations between their rates. Chapter 4 is dedicated for the study of stress state in a body under loadings.

The governing equations for the study of deformation and stress of a body are the global laws presented in Chapter 5. These principles common to all media (such as conservation of mass; the balance of linear momentum, moment of momentum, and energy; and the entropy inequality law) are applied. Kinetics is the study of the static or dynamic equilibrium of forces and moments acting on a body. Thermodynamic principles are concerned with the conservation of energy and relations among heat, mechanical work, and thermodynamic properties of the body. First, we will derive these laws in an integral form, formulated for a finite volume of material in the continuum. Next, we will present the field equations for particles at every point of the studied field.

The relations between stress and strain for a specific material are called *constitutive relations*. We will derive the constitutive relation for elastic bodies in Chapter 6 and for elastic-plastic bodies in Chapter 7. Obtained system of equations will be applied to solve some practical engineering problems.

Several own or cited illustrative examples and exercise problems aim to test and extend the understanding of concepts presented.

The authors of this text have been working together during their stay in the years eighty of last century in the Institute of Technological Researches of the Polish Academy of Sciences, and during the stay of the first author from 2002 to 2012 at the Faculty of Civil Engineering and Architecture of the Lublin University of Technology.

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Chapter 2

The use of tensors

The tensors (mostly tensors of second order) play a very important role in continuum mechanics. All laws of continuum mechanics must be formulated in terms of these quantities that are independent of coordinates. In this chapter we give a brief summary of tensor calculus.

2.1 Index notation

(a) In a three-dimensional space, frequently we denote the axes of the cartesian coordinate systems as x, y, z and the unit vectors as $\mathbf{i}, \mathbf{j}, \mathbf{k}$. For future convenience, it is useful to abbreviate them by using a single component with a generalized index, so we write x_1, x_2, x_3 or

$$x_i \quad (i = 1, 2, 3) \tag{2.1}$$

and we can denote the unit vectors for example as $\mathbf{e}_i$ $(i = 1, 2, 3)$.

Also, the homogeneous linear function can be defined as

$$a_1x_1 + a_2x_2 + a_3x_3 = \sum_{i=1}^{3} a_m x_m = 0 \tag{2.2}$$

where a_m $(m = 1, 2, 3)$ are constants. The set of variables a_i, x_i ... that have only one index is called *object of order one*, and a_1, a_2, a_3 or x_1, x_2, x_3 ... are called its components. In this case the index i ranges from 1 to the dimension of the related space, and during this course the indices have mainly values $1, 2$ and 3.

(b) The homogeneous quadratic function has the form

$$a_{11}(x_1)^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}(x_2)^2 + a_{23}x_2x_3 + \tag{2.3}$$

$$+a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}(x_3)^2 = \sum_{m,n=1}^{3} a_{mn}x_mx_n = 0$$

where a_{mn} are constants. The coefficients of this function have 2 indices. We will call them *objects of order two*. Each object of order two has $3^2 = 9$ components.

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(c) In the same way we can define the objects of order three and four as

$$a_{ijk}, \qquad A_{ijkl} \tag{2.4}$$

which have 3 and 4 indices respectively. Object of order three (in three-dimensional space) has $3^3 = 27$ components and object of order four - $3^4 = 81$ components.

Object a, which has no index is called *object of order zero*. It has $3^0 = 1$ component.

2.2 Summation convention

In the expressions (2.2) and (2.3) we can eliminate the use of the summation symbol $\sum$ by adopting the following convention: *If an index occurs precisely twice in a term of an expression, then it will be understood that we have the summation with respect to that index over its range (in this case from* 1 *to* 3*).* Now the expressions (2.2) and (2.3) can be rewritten as follows

$$a_m x_m, \qquad a_{mn} x_m x_n \tag{2.5}$$

An index that is summed over is called *a dummy index*. Again, this index itself can be freely chosen because of the fact that the particular letter used is not important:

$$a_m x_m = a_i x_i = a_j x_j \quad \text{etc.} \tag{2.6}$$

An index that is not summed over is called *a free index* which can take any value from the set of numbers $1, 2$ and 3. Note that the free index appearing in every term of an equation must be the same. Hence the equation $a_i = b_k$ has no meaning.

2.3 Some operations

(a) Addition and substraction - defined only for objects of the same order

$$a_{ij} \pm b_{ij} = c_{ij}, \qquad a_i \pm b_i = c_i \qquad \text{etc.}$$

(b) Multiplication - can be applied for objects of any order. Multiplication of an object of order m by an object of order n yields object of order $m + n$

$$c_{ijkl} = a_{ij} b_{kl}, \quad c_{ij} = a_i b_j$$

(c) Contraction: Consider object of order four A_{ijkl}, a set of 81 components. Giving two indexes the same letter, say by replacing the j by i, will result in A_{iikl}. Contraction reduces the order of the object by 2, so this object now has only two free indices (k and l), a set of 9 components, each being the sum of three of the original components $A_{iikl} = A_{11kl} + A_{22kl} + A_{33kl}$ (the summation convention applied). This set now is an object of order two. As another example, B_{ii} is the contraction of the object B_{ij} with

$$B_{ii} = B_{11} + B_{22} + B_{33}$$

which now is a scalar called the *trace* of B_{ij}.

2.4 Some special objects

Symmetry and anti-symmetry

$$a_{ij} = a_{ji} \quad - \text{symmetry object (6 independent components)}$$

$$a_{ij} = -a_{ji} \quad - \text{anti-symmetry object (only 3 independent components}$$
$$\text{because } a_{11} = a_{22} = a_{33} = 0)$$

$$A_{ijk} = A_{jki} = A_{kij} = -A_{ikj} = -A_{kji} = -A_{jik}, \text{ absolute anti-symmetry object}$$
$$\text{of order three.}$$

Such object has only one independent component A_{123}. All others are equal to $+A_{123}$ or $-A_{123}$ whether or not the indices permute like $1, 2, 3$. Whenever the values of any two indices coincide, A_{ijk} vanishes, for example $A_{112} = A_{333} = 0$.

Absolute anti-symmetry object for it $A_{123} = +1$ is called *the permutation symbol* and is defined by

$$\epsilon_{ijk} \tag{2.7}$$

$$\epsilon_{123} = 1, \epsilon_{231} = 1, \epsilon_{213} = -1 \qquad \text{etc.}$$

The Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{2.8}$$

Note that because of the summation convention

$$\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

It is easy to verify that $A_{ij}\delta_{jk} = A_{ik}$.

We give without proof the following theorem and lemmas concerning the permutation symbol ϵ_{ijk} and the Kronecker deltar δ_{mn}:

Theorem 1

$$\epsilon_{ijk}\epsilon_{mnp} = \delta_{im}\delta_{jn}\delta_{kp} + \delta_{in}\delta_{jp}\delta_{km} + \delta_{ip}\delta_{jm}\delta_{kn} - \tag{2.9}$$
$$-\delta_{im}\delta_{jp}\delta_{kn} - \delta_{ip}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kp}$$

Lemma 1 *Replacing $p = k$ in (2.9) we have*

$$\epsilon_{ijk}\epsilon_{mnk} = 3\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} + \delta_{in}\delta_{jm} - \delta_{im}\delta_{jn} - \delta_{im}\delta_{jn} - 3\delta_{in}\delta_{jm} \tag{2.10}$$

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

Lemma 2 *Next, putting $j = n$ in (2.10):*

$$\epsilon_{ink}\epsilon_{mnk} = 2\delta_{im} \tag{2.11}$$

Problem 1 *Prove that if A_{ij} is an symmetric object, and B_{ijk}- anti-symmetric object with respect to the indices i, j, which means $A_{ij} = A_{ji}$, $B_{ijk} = -B_{jik}$, then $A_{ij}B_{ijk} = 0$.*

Solution:

$$A_{ij}B_{ijk} = A_{ij}\left(\frac{1}{2}\,B_{ijk} + \frac{1}{2}\,B_{ijk}\right) = A_{ij}\left(\frac{1}{2}\,B_{ijk} - \frac{1}{2}\,B_{jik}\right) = \frac{1}{2}\,A_{ij}B_{ijk} - \frac{1}{2}\,A_{ji}B_{jik} =$$

$$= \frac{1}{2}\,A_{ij}B_{ijk} - \frac{1}{2}\,A_{ij}B_{ijk} = 0 \;\; (Q.E.D.)$$

We have used in the second term the symmetry of $A_{ij} = A_{ji}$, then changed the dummy indices $i \to j; j \to i$.

2.5 Determinant

The determinant is defined as

$$\Delta \equiv \det|a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \tag{2.12}$$

or

$$\Delta = \epsilon_{ijk}\,a_{i1}a_{j2}a_{k3}$$

The right side of this expression is an absolute anti-symmetric object with respect to the indices $1, 2, 3$:

$$\epsilon_{ijk}\,a_{i1}a_{j2}a_{k3} = -\epsilon_{ijk}\,a_{i2}a_{j1}a_{k3} \qquad \text{etc.}$$

hence we can rewrite (2.12) in the form

$$\epsilon_{mnp}\,\Delta = \epsilon_{ijk}\,a_{im}a_{jn}a_{kp} \tag{2.13}$$

Multiplying the above expression by ϵ_{mnp} using (2.11), after putting $m = i$ in it, we get

$$\Delta = \frac{1}{6}\,\epsilon_{ijk}\,\epsilon_{mnp}\,a_{im}a_{jn}a_{kp} \tag{2.14}$$

2.6 Coordinate system and vector algebra

Consider a cartesian coordinate system x_i in the three dimensional space with the unit vectors on these axes $\mathbf{e}_i$, $(i = 1, 2, 3)$. We distinguish two kinds of coordinate systems: the right-handed and the left-handed presented on the Figure 2.1. In this course, we use only *right-handed coordinate system.*

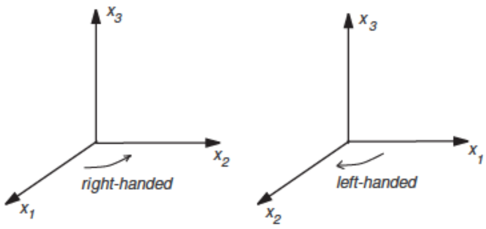

Figure 2.1: Right-handed and left-handed coordinate system

Let $\mathbf{a}$ be any vector. The decomposition of the vector $\mathbf{a}$ on the axes x_i will be:

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a_i\mathbf{e}_i \tag{2.15}$$

where $\mathbf{a}_i$ are components of $\mathbf{a}$. Let $\mathbf{b}$ be another vector, $\mathbf{b} = b_k\mathbf{e}_k$. The *scalar product* of vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\alpha \tag{2.16}$$

where $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_m a_m} = \sqrt{\delta_{ij} a_i a_j}$ stands for the absolute length of the vector $\mathbf{a}$, and α is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$ measured in the plane containing them. From the equation (2.16) we see that if either of the vectors is a unit vector, the scalar product gives the projected length of the other vector in the direction of the unit vector. Hence the scalar product $\mathbf{e}_i \cdot \mathbf{e}_j$ is 1 if $i = j$ and 0 if $i \neq j$:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{2.17}$$

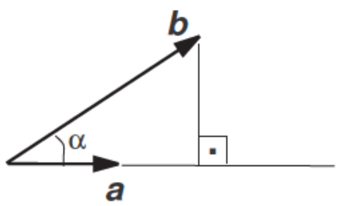

Figure 2.2: Scalar (dot) product

Since the scalar product of vectors is distributive:

$$\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \cdot \mathbf{e}_j) = \delta_{ij} a_i b_j = a_i b_i \tag{2.18}$$

The vector product (also called the cross product) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ satisfying the following three conditions:

1. Its absolute value is equal to the area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$ (see Fig. (2.3)); i.e.,

$$|\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\alpha = S. \tag{2.19}$$

2. It is perpendicular to the plane of the parallelogram; i.e., $\mathbf{c} \perp \mathbf{a}$ and $\mathbf{c} \perp \mathbf{b}$.

3. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a right-handed coordinate system; i.e., the vector $\mathbf{c}$ points to the side from which the sense of the shortest rotation from $\mathbf{a}$ to $\mathbf{b}$ is counterclockwise.

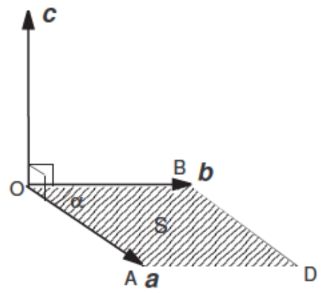

Figure 2.3: The vector product

Using the permutation symbols the three components of the product vector can be written as follows:

$$c_i = \epsilon_{ijk}\, a_j\, b_k \tag{2.20}$$

Remark: If vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, then the parallelogram $OADB$ is degenerate and should be assigned the zero area. Hence the cross product of collinear vectors is defined as the zero vector.

The scalar triple product (also called the mixed or box product) is defined as the dot product of one of the vectors with the cross product of the other two.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_{31} & c_2 & c_3 \end{vmatrix} = \epsilon_{ijk}\, a_i\, b_j\, c_k \tag{2.21}$$

Geometrically, the scalar triple product is the (signed) volume of the parallelepiped defined by the three vectors given. If the scalar triple product is equal to zero, then the three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar, since the "parallelepiped" defined by them would be flat and have no volume. When $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ make a right-handed coordinate system, the scalar triple product is positive.

2.7 Transformation of coordinates

The values of the components of a vector depend on the chosen coordinate system. Often we have to reorient the coordinate system, and the components of the vector change. Suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are unit vectors of two right-handed rectangular Cartesian coordinate system. It is clear that in this case triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be made to coincide with triad $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ through a rigid body rotation. Denote the values of the components of the vector $\mathbf{a}$ in these triads by a_1, a_2, a_3 and a'_1, a'_2, a'_3 respectively. Since the vector is the same, we have:

$$\mathbf{a} = a_i \mathbf{e}_i = a'_j \mathbf{e}'_j \tag{2.22}$$

Multiplying both sides of the above relation respectively by $\mathbf{e}_k$ or $\mathbf{e}'_k$, taking into account (2.17) we obtain:

$$a_k = a_i \delta_{ik} = (\mathbf{e}'_j \cdot \mathbf{e}_k) a'_j \tag{2.23}$$

and

$$a'_k = a'_j \delta_{jk} = (\mathbf{e}_i \cdot \mathbf{e}'_k) a_i \tag{2.24}$$

Denoting the cosine of the angle between $\mathbf{e}'_j$ and $\mathbf{e}_k$ by Q_{jk}, we have respectively from (2.23) and (2.24):

$$a_k = Q_{jk} a'_j \tag{2.25}$$

and

$$a'_k = Q_{ki} a_i \tag{2.26}$$

In matrix notation the equations (2.26) are

$$\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{2.27}$$

Here $(\mathbf{a})'$ denotes the matrix of vector $\mathbf{a}$ with respect to the primed basis $\mathbf{e}'_i$ and $(\mathbf{a})$ denotes the matrix of the same vector with respect to the unprimed basis $\mathbf{e}'_i$. Equations (2.26) represent the transformation law relating the components of the same vector with respect to different Cartesian unit bases. From the definition of Q_{jk} we can write

$$\mathbf{e}'_j = Q_{jk} \mathbf{e}_k, \quad Q_{ji} Q_{jk} = Q_{ij} Q_{kj} = \delta_{ik} \tag{2.28}$$

and as a result of the geometrical interpretation of the vector trip product of the triad $\mathbf{e}'_k$:

$$\det Q_{ij} = +1 \tag{2.29}$$

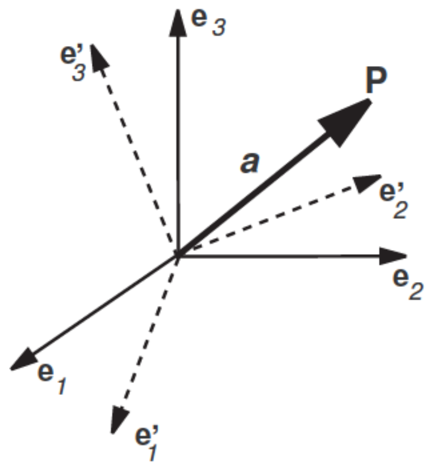

Figure 2.4: Transformation of coordinates

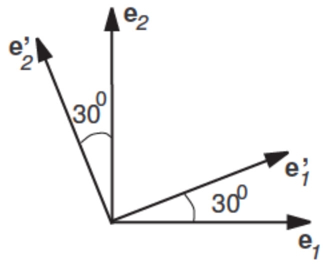

Figure 2.5: For Problem 2

Problem 2 *Let the basic* $\mathbf{e}'_i$ *be obtained by rotating the basis* $\mathbf{e}_j$ *about the* $\mathbf{e}_3$ *axis by* 30^0*, as shown in Figure (2.5). In this figure,* $\mathbf{e}_3$ *and* $\mathbf{e}'_3$ *coincide. Find the* Q_{ij}

Solution:

We can obtain the transformation matrix in two ways:

1. Using the definition of Q_{ij}, we have

$$
\begin{aligned}
Q_{11} &= \cos(\mathbf{e}'_1, \mathbf{e}_1) = \cos 30^0 = \sqrt{3}/2 \\
Q_{12} &= \cos(\mathbf{e}'_1, \mathbf{e}_2) = \cos 60^0 = 1/2 \\
Q_{13} &= \cos(\mathbf{e}'_1, \mathbf{e}_3) = \cos 90^0 = 0 \\
\ldots &= \ldots \\
Q_{33} &= \cos(\mathbf{e}'_3, \mathbf{e}_3) = \cos 0^0 = 1
\end{aligned}
$$

2. It is easier to simply look at Figure(2.5) and decompose each of the $\mathbf{e}'_j$ into its components in the $\mathbf{e}_i$ directions, i.e.,

$$
\begin{aligned}
\mathbf{e}'_1 &= \cos 30^0\mathbf{e}_1 + \cos 60^0\mathbf{e}_2 + 0\cdot\mathbf{e}_3 = (\sqrt{3}/2)\vec{e}_1 + (1/2)\vec{e}_2 \\
\mathbf{e}'_2 &= -\cos 60^0\mathbf{e}_1 + \cos 30^0\mathbf{e}_2 + 0\cdot\mathbf{e}_3 = -(1/2)\mathbf{e}_1 + (\sqrt{3}/2)\mathbf{e}_2 \\
\mathbf{e}'_3 &= \mathbf{e}_3
\end{aligned}
$$

Thus, we have:

$$Q_{ij} = \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.8 Definition of Cartesian Tensors

We have proved that a vector is completely defined by its three components. When we know the components of a vector in the x_i coordinate system, then the new components of such a vector can be calculated using the relation (2.26). Similarly, let the coordinates of point P on the figure (2.4) be x_i on the unprimed coordinate system and x'_k in the primed system. Then:

$$x'_k = Q_{ki} x_i \qquad \text{and} \qquad x_k = Q_{ik} x'_i \tag{2.30}$$

It follows that

$$Q_{ki} = \frac{\partial x'_k}{\partial x_i} = \frac{\partial x_i}{\partial x'_k} \tag{2.31}$$

The transformation (2.26) is valid for any kind of vectors: radius vector, force, velocity etc. and we shall adopt it as *the definition of a vector*, thus replacing the traditional definition of vector as a quantity possessing direction and magnitude. The basic reason is that it can be easily generalized to more complicated physical quantity called *tensors*.

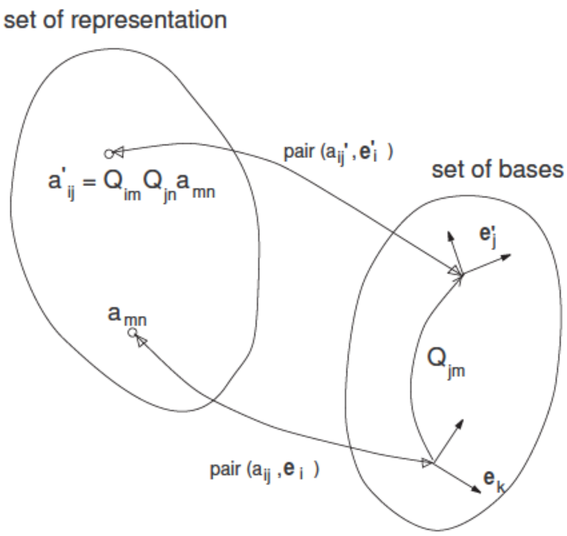

Figure 2.6: Illustration of tensor of rank two

Definition 1 *The pair* $(a, \mathbf{e}_k)$ *determines a zero-order cartesian tensor (or a scalar) if the object* a *is unaffected by the transformation of coordinate system (2.28).*

Definition 2 *The pair $(A_i, \mathbf{e}_k)$ determines a first order cartesian tensor (or a vector) if under the transformation of unit vectors (2.28) object A_i has the new values*

$$A'_j = Q_{ji}\, A_i \tag{2.32}$$

Definition 3 *A pair $(A_{ij}, \mathbf{e}_k)$ determines a cartesian tensor of order two if under the transformation of unit vectors (2.28) the object A_{ij} transforms into the new values*

$$A'_{ij} = Q_{im} Q_{jn} A_{mn} \tag{2.33}$$

The pairs $(A_{ij}, \mathbf{e}_k)$ and $(A'_{ij}, \mathbf{e}'_l)$ represent the same second order tensor. A_{ij} are the values of this tensor in a coordinate system $\mathbf{e}_k$ while A'_{ij} are values in a new coordinate system $\mathbf{e}'_l$.

In matrix notation, the equation (2.33) is:

$$\begin{pmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{pmatrix} \tag{2.34}$$

Definition 4 *The pair $(A_{i...k}, \mathbf{e}_k)$ determines a tensor of n-order if under the transformation of unit vectors (2.28) the object $A_{i...k}$ transforms to*

$$A'_{j...m} = Q_{ji}....Q_{mk}\, A_{i...k} \tag{2.35}$$

both $A_{i..k}$ and $A'_{j..m}$ have n free indices.

Tensors, whose components are the same in all coordinate systems, are called *isotropic tensors* [9]. The permutation symbol is an example of cartesian isotropic tensor of order three, and Kronecker delta is an isotropic Cartesian tensor of second order.

Tensors will be denoted by bold-case letters. We use the summation convention and the following notations, with suitable extensions for tensors of other order.

$$\mathbf{1} \leftrightarrow \delta_{kl} \ \text{ -- the identity tensor} \tag{2.36}$$

$$(\overset{T}{\mathbf{A}})_{kl} = A_{lk} \ \text{ -- transpose of } \mathbf{A} \tag{2.37}$$

$$\overset{-1}{\mathbf{A}} \text{ inverse of } \mathbf{A} \ - \ (\overset{-1}{A}_{kl} A_{lm} = \delta_{km}) \tag{2.38}$$

$$\mathbf{AB} \ \rightarrow \ A_{kl}B_{lm} \text{ or } A_{klmn}B_{mn} \tag{2.39}$$

$$\mathbf{A} \otimes \mathbf{B} \ \rightarrow \ A_k B_l \ \text{ -- the tensor product} \tag{2.40}$$

$$\mathbf{A} \cdot \mathbf{B} \ \rightarrow \ A_k B_k \text{ or } A_{kl}B_{kl} = \text{trace}\,(\mathbf{A}\overset{T}{\mathbf{B}}) \tag{2.41}$$

Now we can rewrite for example the relations between (2.32) and (2.33):

$$\mathbf{A}' = \mathbf{Q}\,\mathbf{A} \qquad \mathbf{A} \text{ is a tensor of first order, vector} \tag{2.42}$$

$$\mathbf{A}' = \mathbf{Q}\,\mathbf{A}\,\overset{T}{\mathbf{Q}} \qquad \mathbf{A} \text{ is a tensor of second order} \tag{2.43}$$

We need to distinguish tensor of second order $\mathbf{T}$ from its matrix of representation T_{ij} in some coordinate system $\mathbf{e}_i$. Using the notation (2.40), the tensor product of two vector $\mathbf{u}$ and $\mathbf{v}$ is a tensor $\mathbf{w}$ with component $w_{ij} = u_i\, v_j$. In matrix form:

$$\begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_1 \;\, v_2 \;\, v_3) = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix} \tag{2.44}$$

The product tensorial $\mathbf{e}_i \otimes \mathbf{e}_j$ of two unit vectors makes a basis for tensors of second order. The components of them are for example

$$(\mathbf{e}_1 \otimes \mathbf{e}_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (\mathbf{e}_1 \otimes \mathbf{e}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dots. \tag{2.45}$$

Thus we can write:

$$\mathbf{T} = T_{ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j) \tag{2.46}$$

The tensor product has a following property for three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c} = \mathbf{a}\,(\mathbf{b} \cdot \mathbf{c}) \;\Rightarrow\; (a_i b_j c_j) \tag{2.47}$$

From (2.47) and (2.46) we have:

$$(\mathbf{T}\,\mathbf{e}_k) = T_{ij}\,(\mathbf{e}_i \otimes \mathbf{e}_j)\,\mathbf{e}_k = T_{ij}\mathbf{e}_i\,\delta_{jk} = T_{ik}\,\mathbf{e}_i \tag{2.48}$$

Multiplying both sides of (2.48) scalar with $\mathbf{e}_m$ we obtain:

$$\mathbf{e}_m \cdot (\mathbf{T}\,\mathbf{e}_k) = \mathbf{e}_m \cdot (T_{ik}\,\mathbf{e}_i) = \delta_{mi}\,T_{ik} = T_{mk} \tag{2.49}$$

or $T_{mk} = \mathbf{e}_m \cdot (\mathbf{T}\,\mathbf{e}_k)$. This formula is valid with respect to any bases used, for example:

$$T'_{mk} = \mathbf{e}'_m \cdot (\mathbf{T}\,\mathbf{e}'_k) \tag{2.50}$$

and is very convenient in case when we only want to calculate some special components of a tensor with respect to some chosen bases.

2.9 Gradient of the scalar field. Normal vector to the surface

Suppose that a scalar f is defined over a region of three-dimensional space x_i $(i = 1, 2, 3)$. The equation

$$f(x_i) = 0 \tag{2.51}$$

defines a surface in this space. Then vector

$$\frac{\partial f}{\partial x_i},$$

called the *gradient* of f, is normal to this surface at point x_i.

The length of this vector can be calculate by

$$\left|\frac{\partial f}{\partial x_i}\right| = \left(\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}\right)^{\frac{1}{2}}$$

Then the unit normal vector is as follows

$$n_i = \frac{\dfrac{\partial f}{\partial x_i}}{\left(\dfrac{\partial f}{\partial x_k}\dfrac{\partial f}{\partial x_k}\right)^{\frac{1}{2}}} \tag{2.52}$$

From now, we use the following notation for partial derivation:

$$\frac{\partial f}{\partial x_i} = f_{,i}\ ;\ \frac{\partial^2 A}{\partial x_i\,\partial x_k} = A_{,ik} \tag{2.53}$$

The expression (2.52) now has the form

$$n_i = \frac{f_{,i}}{(f_{,k}\,f_{,k})^{\frac{1}{2}}} \tag{2.54}$$

Problem 3 *Find the unit normal vector of the surface with equation:*

$$f = a_i x_i - c = 0, \quad a_1 = a_2 = a_3 = 1, \quad c = const$$

Solution:

Calculate the partial derivations of f

$$f_{,i} = \frac{\partial f}{\partial x_i} = a_i$$

then

$$n_i = \frac{a_i}{(a_k a_k)^{\frac{1}{2}}}$$

When $a_1 = a_2 = a_3 = 1$

$$n_1 = n_2 = n_3 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \tag{2.55}$$

$$\mathbf{n}\cdot\mathbf{e}_k = \cos(\mathbf{n}, \mathbf{e}_k) = \frac{1}{\sqrt{3}}$$

2.10 Invariants of a symmetric tensor of second order. Principal directions

Consider a symmetric tensor of second order $\mathbf{T}$, whose representation on bases $\mathbf{e}_i$ is a symmetric matrix $T_{ik} = T_{ki}$. Let n_i be the components on bases $\mathbf{e}_i$ of a unit vector $\mathbf{n}$. This vector determines a direction in the space. We are now looking for a special direction n_i, so that the multiplication $T_{ik}n_k$ is parallel to n_i:

$$\mathbf{Tn} = T\,\mathbf{n} \;\Rightarrow\; T_{ik}n_k = Tn_i \tag{2.56}$$

where T is a coefficient of proportionality. The above relation can be rewriten as:

$$(T_{ik} - T\delta_{ik})\,n_k = 0 \;\text{ with }\; n_i\,n_i = 1 \tag{2.57}$$

In long form:

$$\begin{aligned} (T_{11} - T)n_1 + T_{12}n_2 + T_{13}n_3 &= 0 \\ T_{21}n_1 + (T_{22} - T)n_2 + T_{23}n_3 &= 0 \\ T_{31}n_1 + T_{32}n_2 + (T_{33} - T)n_3 &= 0 \end{aligned} \tag{2.58}$$

Equations (2.58) are a system of linear homogeneous equations in n_1, n_2 and n_3. Obviously, a solution for this system is $n_1 = n_2 = n_3 = 0$. This is known as *the trivial solution*. To find the nontrivial solutions, we note that a system of homogeneous, linear equations admits a nontrivial solution only if the determinant of its coefficients vanishes. That is,

$$\det|T_{ik} - T\delta_{ik}| = 0 \tag{2.59}$$

Expanding the determinant using (2.14) results in a cubic equation in T.

$$T^3 - I_T T^2 + II_T T - III_T = 0 \tag{2.60}$$

Equation (2.60) is called the characteristic equation of $\mathbf{T}$, where

$$\begin{aligned} I_T &= T_{ii} = T_{11} + T_{22} + T_{33} \\ II_T &= \frac{1}{2}\,\epsilon_{ijr}\,\epsilon_{lmr}\,T_{il}\,T_{jm} = \frac{1}{2}\,(\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\,T_{il}\,T_{jm} = \frac{1}{2}\,(T_{ii}T_{mm} - T_{mi}T_{im}) \\ &= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & T_{31} \\ T_{13} & T_{11} \end{vmatrix} \\ III_T &= \frac{1}{6}\,\epsilon_{ijk}\,\epsilon_{lmn}\,T_{il}T_{jm}T_{kn} = \det|T_{ij}| = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} \end{aligned} \tag{2.61}$$

The values I_T, II_T, III_T are called the *basic invariants* of the tensor $\mathbf{T}$. They are independent. Every other invariants of the tensor $\mathbf{T}$ can be expressed in terms of the basic invariants. For example, in plasticity one frequently uses the so-called *octahedral invariant*, defined as follows

$$T^{(0)} = \sqrt{{I_T}^2 - 3\,II_T} = \sqrt{\frac{3}{2}}\sqrt{T_{mi}T_{im} - \frac{1}{3}\,(T_{kk})^2} \tag{2.62}$$

It can be shown that when $\mathbf{T}$ is symmetric, the equation (2.60) has three real roots, denoted by T_1, T_2, T_3. Having these roots, we can obtain from the equations (2.57) three directions $n_j^{(i)}$ $(i = 1,2,3)$, corresponding to the three roots. The numbers T_1, T_2, T_3 are called *principal values* of the tensor $\mathbf{T}$ and the corresponding vectors $n_j^{(i)}$ are called the *principal directions* of the tensor $\mathbf{T}$. These three principal directions are mutually perpendicular

$$\mathbf{n}^{(i)} \cdot \mathbf{n}^{(j)} = \delta_{ij}$$

A right-handed coordinate system can be oriented to line up with the principal directions of the tensor $\mathbf{T}$ and we call it the *principal axis system.* By definition $\mathbf{Tn} = T\,\mathbf{n}$ then using (2.50):

$$T_{11} = \mathbf{n}^{(1)} \cdot (\mathbf{Tn}^{(1)}) = \mathbf{n}^{(1)} \cdot T_1\mathbf{n}^{(1)} = T_1$$

$$T_{22} = \mathbf{n}^{(2)} \cdot (\mathbf{Tn}^{(2)}) = \mathbf{n}^{(2)} \cdot T_2\mathbf{n}^{(2)} = T_2$$

$$T_{33} = \mathbf{n}^{(3)} \cdot (\mathbf{Tn}^{(3)}) = \mathbf{n}^{(3)} \cdot T_3\mathbf{n}^{(3)} = T_3$$

$$T_{12} = \mathbf{n}^{(1)} \cdot (\mathbf{Tn}^{(2)}) = \mathbf{n}^{(1)} \cdot T_2\mathbf{n}^{(2)} = 0...$$

Then the representation of $\mathbf{T}$ in this coordinate system is

$$T_{11} = T_1,\; T_{22} = T_2,\; T_{33} = T_3; \qquad T_{ik} = 0 \text{ for } i \neq k$$

A plane, its normal makes equal angles with each axis of the principal axes system is called the *octahedral plane.*

Problem 4 *Find the principal values and principal directions for the tensor*

$$T_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{pmatrix}$$

Solution:

The characteristic equation gives

$$\det|T_{ik} - T\delta_{ik}| = \begin{vmatrix} 2-T & 0 & 0 \\ 0 & 3-T & 4 \\ 0 & 4 & -3-T \end{vmatrix} = (2-T)(T^2-25) = 0$$

Thus, there are three distinct eigenvalues, $T_1 = 2$, $T_2 = 5$ and $T_3 = -5$.

For $T_1 = 2$, Eqs. (2.58) gives

$$0n_1 = 0;\ n_2 + 4n_3 = 0;\ 4n_2 - 5n_3 = 0;$$

and we also have Eq. (2.57):

$$(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$$

Thus, $n_2 = n_3 = 0$ and $n_1 = \pm 1$ so that the eigenvector corresponding to $\lambda_l = 2$ is

$$\mathbf{n}^{(1)} = \pm \mathbf{e}_1$$

For $T_2 = 5$, we have

$$-3n_1 = 0;\ -2n_2 + 4n_3 = 0;\ 4n_2 - 8n_3 = 0;$$

thus (note the second and third equations are the same),

$$n_1 = 0;\ n_2 = 2n_3;$$

and the unit eigenvectors corresponding to $T_2 = 5$ are

$$\mathbf{n}^{(2)} = \pm \frac{1}{\sqrt{5}} (2\mathbf{e}_2 + \mathbf{e}_3)$$

Similarly for $T_3 = -5$, the unit eigenvectors are

$$\mathbf{n}^{(3)} = \pm \frac{1}{\sqrt{5}} (-\mathbf{e}_2 + 2\mathbf{e}_3)$$

The right-handed principal axis system can be for example the triad: $\mathbf{n}^{(1)} = \mathbf{e}_1$, $\mathbf{n}^{(2)} = \frac{1}{\sqrt{5}} (2\mathbf{e}_2 + \mathbf{e}_3)$ and $\mathbf{n}^{(3)} = \frac{1}{\sqrt{5}} (-\mathbf{e}_2 + 2\mathbf{e}_3)$. In this triad, the representation of tensor $\mathbf{T}$ is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

Problem 5 *Given the tensor*

$$T_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Show that:

a) This tensor has the following principal values $\lambda_1 = 3$; $\lambda_2 = \lambda_3 = 2$ *(obviously the ordering of the eigenvalues is arbitrary)*

b) The principal direction corresponding to $\lambda_1 = 3$ *is* $\pm\mathbf{e}_3$ *and there are actually infinitely many principal directions (any vector perpendicular to* $\mathbf{e}_3$*) corresponding to the double root.*

Problem 6 *Given the tensor*

$$T_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Show that:

a) This tensor has the triple roots $T_1 = T_2 = T_3 = 2$.

b) Any direction is a principal direction.

2.11 The spherical and the deviatoric part of a tensor

Every tensor of second order can be divided into two parts: the spherical and the deviatoric one.

$$T_{ij} = \frac{1}{3}\, I_T\, \delta_{ij} + t_{ij} \tag{2.63}$$

The spherical part of a tensor is defined as

$$\frac{1}{3}\, I_T\, \delta_{ij} \tag{2.64}$$

and

$$t_{ij} = T_{ij} - \frac{1}{3}\, I_T\, \delta_{ij} \tag{2.65}$$

is the *deviatoric part* of the tensor **T**. The deviator t_{ij} has only 5 components independent because its first invariant

$$t_{ii} = I_t = 0 \tag{2.66}$$

Hence, deviator has only two non-zero basic invariants II_t, III_t.

Problem 7 *Show that the octahedral invariant of the tensor* T_{ij} *is equal to the octahedral invariant of his deviator* t_{ij}.

Solution:

Since $I_t = 0$ then from (2.62) the octahedral invariant of the deviator is

$$t^{(0)} = \sqrt{-III_t} = \sqrt{\frac{3}{2}\, t_{mi}\, t_{im}} = \sqrt{\frac{3}{2}}\, \sqrt{\left(T_{mi} - \frac{1}{3}\, I_T\, \delta_{mi}\right)\left(T_{im} - \frac{1}{3}\, I_T\, \delta_{im}\right)} = \\ = \sqrt{\frac{3}{2}}\, \sqrt{T_{mi} T_{im} - \frac{1}{3}\, {I_T}^2} = T^{(0)} \tag{2.67}$$

2.12 The Gauss Theorem

We give without proof the Gauss Theorem. It asserts a remarkable connection between surface integrals and volume integrals. If V is a volume bounded by the closed surface S and $\mathbf{A}$ a vector field that possesses continuous derivatives (and is singled valued in V). Divergence of the vector field $\mathbf{A}$, denoted by div$\mathbf{A}$ is a scalar defined by:

$$\mathrm{div}\mathbf{A} = \frac{\partial \mathrm{A}_1}{\partial \mathrm{x}_1} + \frac{\partial \mathrm{A}_2}{\partial \mathrm{x}_2} + \frac{\partial \mathrm{A}_3}{\partial \mathrm{x}_3} = \mathrm{A}_{\mathrm{i,i}} \tag{2.68}$$

The Gauss Theorem:

$$\int_V \mathrm{div}\mathbf{A}\, \mathrm{dV} = \int_{\mathrm{S}} \mathbf{A} \cdot \mathbf{n}\, \mathrm{dS} \tag{2.69}$$

where $\mathbf{n}$ is the outward pointing unit normal vector to S. The integral in the right side of (2.69) is called the *flux of the vector field* $\mathbf{A}$ through the surface S. The index form of (2.69) is

$$\int_S x_i n_i\, dS = \int_A A_i n_i\, dV \tag{2.70}$$

we have used here the notation for partial derivation (2.53). Note that we may extend this result to the case where $\mathbf{A}$ is a tensor field with the same proviso's.

Problem 8 *Prove that:*

$$\int_S \mathbf{x} \cdot \mathbf{n}\, dS = \int_S x_i n_i\, dS = 3V \tag{2.71}$$

Solution:

A useful application of the Gauss theorem involves the computation of the volume of a solid. Consider the integral

$$\int_S x_i n_i\, dS$$

where $\mathbf{x}$ is the position vector and all other quantities have their meanings as above. The divergence theorem states that

$$\int_S x_i n_i\, dS = \int_V x_{i,i}\, dV = \int_V \left(\frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} \right) dV = \int_V (1{+}1{+}1)\, dV = 3V (Q.E.D.)$$

For example, in the case of a ball (sphere) of radius R, $\mathbf{x} \cdot \mathbf{n} = |\mathbf{x}||\mathbf{n}| \cos 0^0 = R$ and relation (2.69) gives

$$R \int_S dS = RS = 3V$$

The surface area of a sphere is $S = 4\pi R^2$; $4\pi R^3 = 3V$, then $V = (4/3)\pi R^3$.

Chapter 3

Strain

3.1 Deformation gradient and displacement

We look for a tool to describe the deformation of a solid. At time t_0 the solid stands in an initial configuration B_0. In this configuration, a particle P_0 is located by its location vector $\mathbf{X}$ with components X_i in a Cartesian coordinate system with orthonormal basis $\mathbf{e}_i$. As time passes, the particles of this solid move to their relative positions. At time t the solid deforms and stands in the current configuration B. The particle P_0 whose initial location vector was $\mathbf{X}$ can be located by its current location vector $\mathbf{x}$, point P. The components of $\mathbf{x}$ are $x_i = x_i(X_k, t)$. We write:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \text{ or in index form } x_i = x_i(X_1, X_2, X_3, t) \tag{3.1}$$

Eq. (3.1) is said to define a *motion* for a solid; these equations describe the path

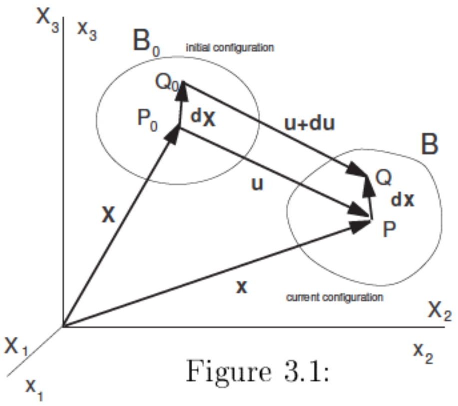

Figure 3.1:

line for every particle of the solid. We suppose that the transformation $P_0 \leftrightarrow P$ is single value and continuous. It is necessary that Jacobian matrix

$$J = \left|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right| = \left|\frac{\partial x_i}{\partial X_j}\right| > 0 \tag{3.2}$$

and we can find the inverse function of (3.1)

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \text{ or in index form } X_i = X_i(x_1, x_2, x_3, t) \tag{3.3}$$

Consider now in the initial configuration an infinitesimal vector $d\mathbf{X}$, which links the solid particle P_0 located at $\mathbf{X}$ to the nearby particle Q_0 located at $\mathbf{X} + d\mathbf{X}$. After deformation $d\mathbf{X}$ becomes $d\mathbf{x}$ linking the same solid particles, which are now located at P and Q as shown in Figure (3.1). Vectors $d\mathbf{X}$ and $d\mathbf{x}$ are material vectors because they consist of the same solid particles. Vector $d\mathbf{x}$ can be obtained from $d\mathbf{X}$ by differentiating (3.1). We obtain:

$$d\mathbf{x} = \mathbf{x}(\mathbf{X} + d\mathbf{X}, t) - \mathbf{x}(\mathbf{X}, t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X} \;;\; dx_i = \frac{\partial x_i}{\partial X_j} dX_j \tag{3.4}$$

or

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X} \;;\; F_{ij} = \frac{\partial x_i}{\partial X_j} dX_j \tag{3.5}$$

$\mathbf{F}$ is named deformation gradient transporting any material vector $d\mathbf{X}$ onto its deformed $d\mathbf{x}$.

The vector $\mathbf{u}(\mathbf{X}, t)$ is called the displacement of the particle whose initial and current positions are P_0 and P respectively:

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \quad \text{or} \quad u_i = x_i - X_i \tag{3.6}$$

From Equations (3.5) and (3.6) the deformation gradient $\mathbf{F}$ can be expressed as a function of the displacement vector $\mathbf{u}$ according to

$$\mathbf{F} = \mathbf{1} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \;;\; F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \tag{3.7}$$

3.2 Material and spatial description

When a solid is in motion, its properties like temperature, stress tensor (to be defined in the next chapter) may change with time. We can describe these changes by one of two ways:

1. Following the particles, i.e., we express these properties as functions of the particles (identified by the material coordinates $X_1; X_2; X_3$ and time t. Such a description is known as the *material description*. Other names for it are the *Lagrangian description* and the *reference description*.

2. Observing the changes at fixed locations, i.e., we express theses properties as functions of fixed position $x_1; x_2; x_3$ and time t. Such a description is known as a *spatial description* or *Eulerian description*. The spatial coordinates x_i of a particle at any time t are related to the material coordinates X_j of the particle by Eq. of motion (3.1). That is, if the motion is known, one description can be obtained from the other.

Note that in spatial description, what is described (or measured) is the change of quantities at a fixed location as a function of time because spatial positions are occupied by different particles at different times.

Problem 9 *Given the function of motion in Lagrangean description:*

$$\begin{cases} x_1 &= X_1 + aX_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{cases} \tag{3.8}$$

Find the spatial description, the strain tensors and the displacement in Lagrangian and Eulerian descriptions.

Solution:

The inverse equations of (3.8) give the spatial description:

$$\begin{cases} X_1 &= x_1 - ax_2 \\ X_2 &= x_2 \\ X_3 &= x_3 \end{cases} \tag{3.9}$$

The Jacobian matrix is:

$$J = \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \tag{3.10}$$

Then we find the Lagrangian strain tensor:

$$E_{jk} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} - \delta_{jk}\right) = \begin{pmatrix} 0 & a/2 & 0 \\ a/2 & a^2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.11}$$

and the Eulerian strain tensor:

$$E^*_{jk} = \frac{1}{2}\left(\delta_{jk} - \frac{\partial X_i}{\partial x_j}\frac{\partial X_i}{\partial x_k}\right) = \begin{pmatrix} 0 & a/2 & 0 \\ a/2 & -a^2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.12}$$

The displacement in Lagrangian and Eulerian descriptions:

$$\begin{cases} u_1 &= aX_2 \\ u_2 &= 0 \\ u_3 &= 0 \end{cases} \qquad \begin{cases} u_1 &= ax_2 \\ u_2 &= 0 \\ u_3 &= 0 \end{cases} \tag{3.13}$$

Problem 10 *Given the function of motion in Lagrangian description:*

$$\begin{aligned} x_1 &= X_1 + X_2(e^t - 1) \\ x_2 &= X_1(e^{-t} - 1) + X_2 \\ x_3 &= X_3 \end{aligned}$$

Find the inverse function and the Lagrangian $\mathbf{u}(\mathbf{X}, t)$ *and Eulerian* $\mathbf{u}(\mathbf{x}, t)$ *displacement functions.*

Answer:

$$\begin{aligned} X_1 &= \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}} \\ X_2 &= \frac{x_1(e^t - 1) - x_2}{1 - e^t - e^{-t}} \\ X_3 &= x_3 \end{aligned}$$

$$\begin{aligned} u_1 &= X_2(e^t - 1) \\ u_2 &= X_1(e^{-t} - 1) \\ u_3 &= 0 \end{aligned}$$

$$\begin{aligned} u_1 &= x_1 - \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}} \\ u_2 &= x_2 - \frac{x_1(e^t - 1) - x_2}{1 - e^t - e^{-t}} \\ u_3 &= 0 \end{aligned}$$

3.3 Measure of deformation. Strain tensor

Deformation induces changes in the length of material vectors and the angle they form. Calculate the square of length of material vector $d\mathbf{X}$ at time t_0 and vector $d\mathbf{x}$ at time t:

$$|P_0Q_0|^2 = dS^2 = dX_1^2 + dX_2^2 + dX_3^2 = dX_j dX_j = \delta_{jk}\, dX_j dX_k = d\mathbf{X} \cdot d\mathbf{X} \quad (3.14)$$

$$|PQ|^2 = ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i = d\mathbf{x} \cdot d\mathbf{x} \quad (3.15)$$

and using (3.5) we have:

$$ds^2 - dS^2 = \frac{\partial x_i}{\partial X_j}\, dX_j\, \frac{\partial x_i}{\partial X_k}\, dX_k - \delta_{jk}\, dX_j dX_k = 2\, E_{jk}\, dX_j dX_k = d\mathbf{X}\,(2\,\mathbf{E})\, d\mathbf{X} \quad (3.16)$$

where:

$$E_{jk} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} - \delta_{jk}\right) = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right) \quad (3.17)$$

Tensor $\mathbf{E}$ is called *Lagrangian* or *Green's strain* tensor (functions of lagrangian coordinates $\mathbf{X}$). The superscript symbol "T" denotes the transpose tensor. Tensor $\mathbf{E}$ is symmetric.

In terms of displacement vector, we can write:

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ik} \quad (3.18)$$

and Eq (3.17) now takes the form:

$$E_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_k} + \frac{\partial u_k}{\partial X_j} + \frac{\partial u_i}{\partial X_j}\frac{\partial u_i}{\partial X_k}\right) \quad (3.19)$$

or

$$\mathbf{E} = \frac{1}{2}\left(\mathrm{grad}\mathbf{u} + \mathrm{grad}^T\mathbf{u} + \mathrm{grad}^T\mathbf{u}\cdot\mathrm{grad}\mathbf{u}\right) \tag{3.20}$$

where grad**u** is a second-order tensor known as the *displacement gradient*. The matrix of grad**u** with respect to rectangular Cartesian coordinates **X** is:

$$\mathrm{grad}\mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{pmatrix} \tag{3.21}$$

When using the Eulerian coordinate, we have the symmetric Eulerian strain tensor, functions of spatial coordinates **x**:

$$E^*_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right) \tag{3.22}$$

3.4 Geometrical meaning of the components of the strain tensor

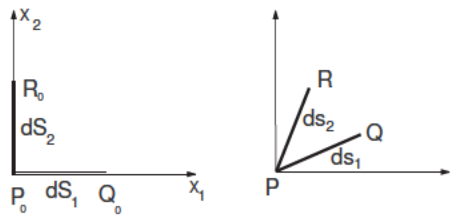

Figure 3.2:

Calculate the unit elongation (i.e., increase in length per unit original length) of PQ:

$$\frac{|\mathbf{PQ}| - |\mathbf{P_0Q_0}|}{|\mathbf{P_0Q_0}|} = \frac{ds - dS}{dS} \equiv \lambda_{PQ} \tag{3.23}$$

When $\lambda_{PQ} > 0$ we have elongation, and when $\lambda_{PQ} < 0$ - reduction. Denoting by $\nu_i = dX_i/dS$, we have

$$\frac{(ds - dS)(ds + dS)}{dS\,dS} = \frac{(ds - dS)(ds - dS + 2\,dS)}{dS\,dS} = 2E_{ik}\frac{dX_i}{dS}\frac{dX_k}{dS} \tag{3.24}$$

and

$$\lambda_{PQ}(\lambda_{PQ} + 2) = 2\,E_{ik}\nu_i\nu_k \tag{3.25}$$

Consider an element dS laying on OX_1 in the initial configuration, then $\nu_j = \delta_{1j}$, $(\nu_1 = 1, \nu_2 = \nu_3 = 0)$. From (3.25):

$$\lambda_{11}(\lambda_{11} + 2) = 2\,E_{11} \quad \Rightarrow \quad \lambda_{11} = \sqrt{1 + 2E_{11}} - 1 \tag{3.26}$$

hence the unit elongation of element $dS = dX_1$ that was in the X_1 direction in the reference state depends on component E_{11}.

Let P_0Q_0 and P_0R_0 be two unit vectors perpendicular to each other that were on the direction X_1 and X_2 in the reference state, see Figure (3.2):

$$P_0Q_0 \parallel X_1, \quad \lambda_{PQ} = \lambda_{11}; \quad \boldsymbol{\nu}^1 = (1,0,0), \quad \nu_i^1 = \delta_{1i}$$

$$P_0R_0 \parallel X_2, \quad \lambda_{PR} = \lambda_{22}; \quad \boldsymbol{\nu}^2 = (0,1,0), \quad \nu_i^2 = \delta_{2i}$$

$$\nu_i = \frac{dX_i}{dS}, \qquad \nu_i^* = \frac{dx_i}{ds}$$

Because

$$x_i = u_i + X_i \quad \Rightarrow \quad dx_i = \left(\frac{\partial u_i}{\partial X_k} + \delta_{ik}\right) dX_k$$

then

$$\nu_i^* = \frac{\left(\dfrac{\partial u_i}{\partial X_k} + \delta_{ik}\right)}{ds} dX_k = \left(\frac{\partial u_i}{\partial X_k} + \delta_{ik}\right) \frac{dX_k}{dS} \frac{dS}{ds}$$

or

$$\nu_i^* = \frac{\left(\dfrac{\partial u_i}{\partial X_k} + \delta_{ik}\right)}{ds} dX_k = \left(\frac{\partial u_i}{\partial X_k} + \delta_{ik}\right) \frac{\nu_k}{1 + \lambda}$$

Due to motion, P_0Q_0 and P_0R_0 become PQ and PR respectively. Let the angle between the two deformed vectors PQ and PR be denoted by φ_{12}^*:

$$\boldsymbol{\nu}^*_{(\mathbf{1})} \cdot \boldsymbol{\nu}^*_{(\mathbf{2})} = \cos\varphi_{12}^* = \frac{2E_{12}}{\sqrt{(1 + 2E_{11})(1 + 2E_{22})}} \tag{3.27}$$

so the change of the angle between two vectors that were on the directions X_1 and X_2 in the initial state, depends on E_{11}, E_{22} and E_{12}.

Hence, six components of the strain tensor E_{ij} describe the deformation of the body. When $E_{ij} = 0$, then $\lambda_{PQ} = 0$ and $\varphi^* = 0 \Rightarrow$ there are no deformations.

Problem 11 *Given the following displacement components [2]:*

$$u_1 = kX_2^2; \; u_2 = u_3 = 0 \tag{3.28}$$

(a) Sketch the deformed shape of the unit square $OA_0B_0C_0$ shown in Figure 3.3.

(b) Calculate the Lagrangian strain tensor.

(c) Find the angle between deformed vectors (i.e., $d\mathbf{x}_1$ and $d\mathbf{x}_2$) of the material elements $d\mathbf{X}_1$ and $d\mathbf{X}_2$, which were at the point C.

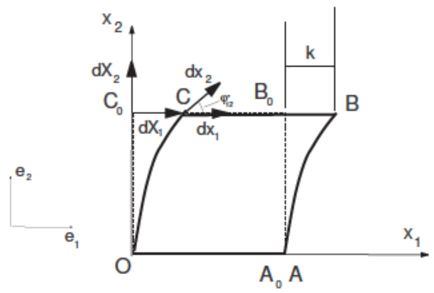

Figure 3.3:

Solution:

a) For particle O the initial coordinate $X_1 = 0, X_2 = 0, X_3 = 0$. From Eq. (3.28) $u_1 = u_2 = u_3 = 0$, so this particle is not displaced. The particle A is also not displaced because $X_1 = 1, X_2 = 0, X_3 = 0$. For the material line C_0B_0; $X_2 = 1; u_1 = k; u_2 = u_3 = 0$, the line is displaced by k units to the right. For the material line OC_0 and A_0B_0, $u_1 = kX_2^2; u_2 = u_3 = 0$, each line becomes parabolic in shape. Thus, the deformed shape is given by $OABC$ shown in Figure 3.3.

b) Using (3.20) we obtain:

$$E_{ij} = \begin{pmatrix} 0 & kX_2 & 0 \\ kX_2 & 2k^2X_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

c) At point C for $X_1 = 0$; $X_2 = 1$; $X_3 = 0$ tensor strain has the form

$$E_{ij} = \begin{pmatrix} 0 & k & 0 \\ k & 2k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For the element $d\mathbf{X}_1$, the unit elongation is $E_{11} = 0$. For the element $d\mathbf{X}_2$, the unit elongation is $E_{22} = 2k^2$. As $E_{12} = k$, from the Eq. (3.27):

$$\cos\varphi_{12}^* = \frac{2E_{12}}{\sqrt{(1+2E_{11})(1+2E_{22})}} = \frac{2k}{\sqrt{1+4k^2}}$$

Since the strain tensor $\mathbf{E}$ is symmetric, there exists at least three mutually perpendicular principal directions $\mathbf{n}_1; \mathbf{n}_2; \mathbf{n}_3$ with respect to which the matrix of E_{ij} is diagonal (see Section 2.10). That is,

$$E_{ij} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}$$

Geometrically, this means that infinitesimal line elements in the directions of $\mathbf{n}_1; \mathbf{n}_2; \mathbf{n}_3$ remain mutually perpendicular after deformation. The unit elongations along the principal directions (i.e., $E_1; E_2; E_3$) are the principal values of $\mathbf{E}$, or *principal strains*. The principal strains are to be found from the characteristic equation of $\mathbf{E}$ (see (2.60)), i.e.,

$$E^3 - I_E E^2 + II_E E - III_E = 0 \tag{3.29}$$

where (see Eq.(2.61))

$$\begin{aligned} I_E &= E_{11} + E_{22} + E_{33} \\ II_E &= \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{33} & E_{31} \\ E_{13} & E_{11} \end{vmatrix} \\ III_E &= \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{vmatrix} \end{aligned} \tag{3.30}$$

The coefficients $I_E; II_E$ and III_E are called the principal invariants of the strain tensor.

3.5 Material derivative. Velocity

The time rate of change of a quantity (such as temperature or velocity or stress tensor) of a material particle is known as a *material derivative*. We shall denote the material derivative by a dot over this quantity. For example, if the displacement is a function of Lagrangian coordinates $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$. For fixed $\boldsymbol{X}$, that means for a particular particle, the velocity is the time partial derivation:

$$\mathbf{v} = \dot{\mathbf{u}} = \frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t} \tag{3.31}$$

When a spatial description of the displacement is used, we have $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, where x_i, the coordinates of the present positions of material particles at time t are related to the material coordinates by the known motion $x_i = x_i(X_1, X_2, X_3, t)$. Then,

$$\mathbf{v} = \dot{\mathbf{u}} = \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} \tag{3.32}$$

or in index form:

$$v_i = \frac{\partial u_i(\mathbf{x}, t)}{\partial t} + \frac{\partial u_k}{\partial x_k} \frac{\mathrm{d}x_k}{\mathrm{d}t}, \qquad \mathbf{v} = \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \mathbf{v} \tag{3.33}$$

In the same way we can calculate the material derivative of other quantities. For example, acceleration $\mathbf{a}$ is the material derivative of velocity in Eulerian coordinates:

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial \mathbf{x}} \cdot \mathbf{v} \tag{3.34}$$

Problem 12 *Given the displacement field (3.13) where $a = \sin t$:*

$$\begin{cases} u_1 = (\sin t)X_2 \\ u_2 = 0 \\ u_3 = 0 \end{cases} \qquad \begin{cases} u_1 = (\sin t)x_2 \\ u_2 = 0 \\ u_3 = 0 \end{cases} \tag{3.35}$$

Find the velocity.

Solution:

In initial coordinates $v_i = \dfrac{\partial u_i}{\partial t}$, then:

$$\begin{cases} v_1 = (\cos t)X_2 \\ v_2 = 0 \\ v_3 = 0 \end{cases} \tag{3.36}$$

and we calculate the material derivative in spatial coordinates:

$$\begin{cases} v_1 = (\cos t)x_2 + t^2 v_2 \\ v_2 = 0 \\ v_3 = 0 \end{cases} \tag{3.37}$$

From these equations we obtain:

$$\begin{cases} v_1 = (\cos t)x_2 \\ v_2 = 0 \\ v_3 = 0 \end{cases} \tag{3.38}$$

Problem 13 *The motion of a continuum is given in the form [2]:*

$$x_1 = X_1 + ktX_2;\ x_2 = (1+kt)X_2;\ x_3 = X_3 \tag{3.39}$$

If the temperature field is given by the spatial description

$$T = \alpha(x_1 + x_2) \tag{3.40}$$

then:

(a) Find the material description of temperature and (b) obtain the velocity and the rate of change of temperature for particular material particles and express the answer in both a material and a spatial description.

Solution:

(a) Substituting Eq. (3.39) into Eq. (3.40), we obtain the material description for the temperature,

$$T = \alpha[X_1 + ktX_2 + (1+kt)X_2] = \alpha X_1 + \alpha(1+2kt)X_2 \tag{3.41}$$

(b) From Eq. (3.39)

Since the displacement is given in Lagrangian coordinates, the velocity is the partial derivative par rapport t:

$$v_1 = kX_2;\ v_2 = kX_2;\ v_3 = 0 \tag{3.42}$$

and the rate of change of temperature in Lagrangian coordinates:

$$\dot{T} = 2\alpha k X_2 \tag{3.43}$$

From equations (3.39) we obtain:

$$X_2 = \frac{x_2}{1+kt} \tag{3.44}$$

By putting (3.44) in (3.42) and (3.43) we have the spatial description for velocity and rate of temperature:

$$v_1 = v_2 = k\,\frac{x_2}{1+kt};\ v_3 = 0$$

$$\dot{T} = 2\alpha\, k\,\frac{x_2}{1+kt}$$

Although the Eulerian temperature field is independent of time, we observe the change in time of temperature of each particle, because it moves in space.

3.6 The rate of deformation tensor

Calculate the material derivative of the material element $d\mathbf{x}$. From (3.4)-(3.5) we have:

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\, d\mathbf{X} = \mathbf{F}\, d\mathbf{X} \tag{3.45}$$

then:

$$(d\mathbf{x})^{\cdot} = d\mathbf{v} = \dot{\mathbf{F}}\, d\mathbf{X} = \dot{\mathbf{F}}\mathbf{F}^{-1}\, d\mathbf{x} \tag{3.46}$$

or

$$d\mathbf{v} = \mathrm{grad}\mathbf{v}\, d\mathbf{x} \tag{3.47}$$

where

$$\mathrm{grad}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \tag{3.48}$$

This leads to the definition of velocity gradient:

$$\mathbf{L} = \mathrm{grad}\mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{3.49}$$

In Cartesian coordinates:

$$(\mathrm{grad}\mathbf{v})_{ij} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{3.50}$$

Decomposing $\mathbf{L}$ into symmetry and antisymmetry parts:

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) \Rightarrow \text{the rate of deformation tensor} \tag{3.51}$$

$$\mathbf{\Omega} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right) \Rightarrow \text{the spin tensor} \tag{3.52}$$

With respect to Cartesian rectangular system of coordinates:

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{3.53}$$

$$\Omega_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) & 0 \end{pmatrix} \tag{3.54}$$

Tensor $\mathbf{D}$ is called *rate of deformation tensor* and describes the rate of change of length and the rate of change of direction of the material element $d\mathbf{x}$, while *the spin tensor* $\mathbf{\Omega}$ only rotates this element (without changing its length).

3.7 Infinitesimal deformation

Assume that:

$$\left|\frac{\partial u_i}{\partial X_j}\right| \ll 1 \tag{3.55}$$

then from (3.6) $\partial/\partial x_j \simeq \partial/\partial X_j$, the current and the initial configurations can eventually be merged $x_i \simeq X_i$. The Lagrangian and Eulerian strain tensors coincide:

$$E_{ij} \simeq E^*_{ij} \simeq \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.56}$$

In Cartesian system of coordinate:

$$\varepsilon_{ij} = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_2}{\partial x_3}\right) & \dfrac{\partial u_3}{\partial x_3} \end{pmatrix} \tag{3.57}$$

Tensor $\boldsymbol{\varepsilon}$ is known as the *infinitesimal strain tensor*. Components ε_{11}, ε_{22} and ε_{33} are called *normal strains*, while ε_{ij} $(i \neq j)$ are called *shear strains*.

Since ε_{ij} has the same order of magnitude as $\left|\dfrac{\partial u_i}{\partial X_j}\right|$, then when $\left|\dfrac{\partial u_i}{\partial X_j}\right| \ll 1$, the infinitesimal strain tensor $|\varepsilon_{ij}| \ll 1$. On the other hand, when $|\varepsilon_{ij}|$ are very small, for example in the case of movement of rigid body $|\varepsilon_{ij}| = 0$, $\left|\dfrac{\partial u_i}{\partial X_j}\right|$ can have any order of magnitude.

The antisymmetric tensor

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{3.58}$$

is known as the *infinitesimal rotation tensor*.

Note: We can write the equations (3.22) in the form:

$$E_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right) = \varepsilon_{jk} + \frac{1}{2}(\varepsilon_{ij} + \omega_{ij})(\varepsilon_{ki} - \omega_{ki})$$

so $E_{jk} \approx \varepsilon_{jk}$ only when ε_{jk} and ω_{jk} are infinitesimal.

Problem 14 *Let us consider the finite rotation of a bar (see Figure 3.4). We have:*

$$\begin{aligned} X &= R\cos\vartheta_0 & \qquad x &= R\cos(\vartheta + \vartheta_0) \\ Y &= R\sin\vartheta_0 & \qquad y &= R\sin(\vartheta + \vartheta_0) \end{aligned}$$

The displacement vector in material description:

$$u(X,Y) = x - X = R\cos(\vartheta + \vartheta_0) - R\cos\vartheta_0 = X(\cos\vartheta - 1) - Y\sin\vartheta$$

$$v(X,Y) = y - Y = X\sin\vartheta + Y(\cos\vartheta - 1)$$

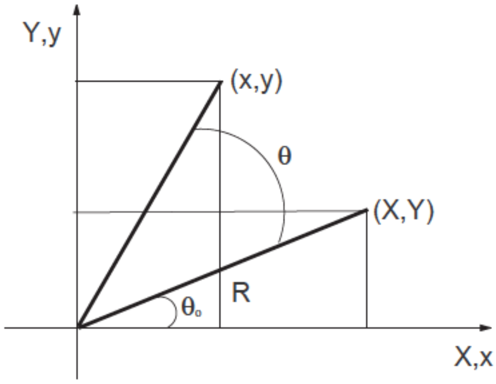

Figure 3.4:

The infinitesimal strain tensor has the form:

$$\begin{aligned}\varepsilon_{xx} &= \cos\vartheta - 1\\ \varepsilon_{yy} &= \cos\vartheta - 1\\ \varepsilon_{xy} &= 0\end{aligned}$$

and cannot describe the rigid rotation of the bar properly (there are no deformations here, but $\varepsilon_{xx} = \varepsilon_{yy} \neq 0$). If we calculate the Lagrangean tensor of deformation, then we see that it is equal zero, for example:

$$E_{xx} = \frac{\partial u}{\partial X} + (1/2)\left[\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2\right] = (\cos\vartheta - 1) + (1/2)\left[(\cos\vartheta - 1)^2 + \sin^2\vartheta\right] = 0$$

This is important in the case of deformation of bars or shells, when deformations are small while the angle of rotation is finite.

The geometrical interpretation of components of the infinitesimal strain and rotation tensors are presented in figure (3.5) in two-dimensional case. In this case, the relations (3.26) and (3.27) take the forms:

$$\lambda_{11} = \varepsilon_{11} \ ; \ \lambda_{22} = \varepsilon_{22} \tag{3.59}$$

that means ε_{11} and ε_{22} describe the changes in length per unit of length in the directions of the coordinate axes x_1 and x_2 respectively. The decrease in angle between the positive directions of the two coordinate line elements is described by the component ε_{12}:

$$\cos\varphi^*_{12} = 2\varepsilon_{12} \tag{3.60}$$

and the rotation as the rigid body is described ω_{12}.

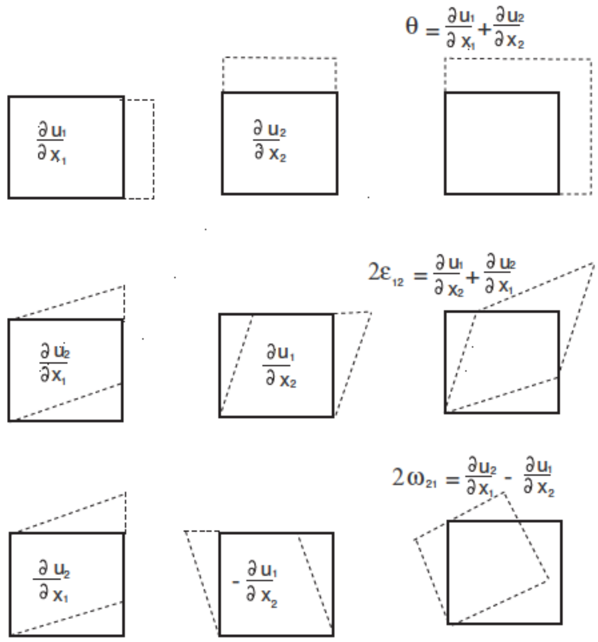

Figure 3.5: Geometrical interpretation of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$

3.8 Principal directions of the infinitesimal strain tensor

Tensor $\boldsymbol{\varepsilon}$ is a symmetric tensor of order two. We can find the principal directions from the equations:

$$(\varepsilon_{ij} - \varepsilon\, \delta_{ij})\, n_j = 0 \tag{3.61}$$

$$\varepsilon^3 - I_\varepsilon \varepsilon^2 + II_\varepsilon \varepsilon - III_\varepsilon = 0 \tag{3.62}$$

where:

$$\begin{aligned} I_\varepsilon &= \varepsilon_{kk} \\ II_\varepsilon &= 1/2(\,\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ji}) \\ III_\varepsilon &= \det \varepsilon_{\mathrm{ij}} = (1/6)\, \epsilon_{\mathrm{ijk}}\, \epsilon_{\mathrm{mnp}}\, \varepsilon_{\mathrm{im}}\, \varepsilon_{\mathrm{jn}}\, \varepsilon_{\mathrm{kp}} \end{aligned} \tag{3.63}$$

From the characteristic equation (3.62) we find three principal strains ε_1, ε_2, ε_3. We can show that the first scalar invariant of the infinitesimal strain tensor has a simple geometric meaning. Consider a rectangular parallelepiped based on three material fibers along the principal directions of tensor $\boldsymbol{\varepsilon}$. Let dS_1, dS_2 and dS_3 be the boxes before deformation, so the volume is $dV_0 = dS_1\, dS_2\, dS_3$. These boxes have been elongated and have the lengths $(1 + \varepsilon_1)dS_1, (1 + \varepsilon_2)dS_2$ and $(1 + \varepsilon_3)dS_1$, so the volume after deformations is $dV = (1 + \varepsilon_1)\,(1 + \varepsilon_2)\,(1 + \varepsilon_3)\, dS_1\, dS_2\, dS_3$. Then

$$\frac{dV - dV_0}{dV_0} \simeq I_\varepsilon = \varepsilon_{kk} = \mathrm{div}\mathbf{u} \tag{3.64}$$

It means that the first scalar invariant of the infinitesimal strain tensor is equal to the unit volume change known as the *dilatation*. In Cartesian coordinates:

$$I_\varepsilon = \mathrm{div}\mathbf{u} = \frac{\partial \mathrm{u}_1}{\partial \mathrm{x}_1} + \frac{\partial \mathrm{u}_2}{\partial \mathrm{x}_2} + \frac{\partial \mathrm{u}_3}{\partial \mathrm{x}_3} \tag{3.65}$$

It is often useful to extract the deviatoric part (see (2.65)) accounting for the material distortion only, from the deformation tensor $\boldsymbol{\varepsilon}$. The resulting tensor $\mathbf{e}$ is the deviatoric strain tensor, expressed by

$$\mathbf{e} = \boldsymbol{\varepsilon} - \frac{1}{3} I_\varepsilon \mathbf{1} \tag{3.66}$$

or in index form:

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \delta_{ij} \tag{3.67}$$

The strain deviator e_{ij} relates to the change in shape, while ε_{kk} relates to the volume change of the element.

3.9 The compatibility equations for the infinitesimal strain tensor

Six components of the infinitesimal strain tensor ε_{ij} depend on three components of the displacement vector u_i (3.56), so we have six functions to find three functions u_i, then in general, we can not find a single-valued solution. Hence, the components of the strain tensor must satisfy some conditions. They are called *compatibility conditions* or *compatibility equations*.

1. From (3.57), calculate the following derivatives:

$$\begin{aligned}
\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} &= \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} \\
\frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} &= \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \\
2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} &= \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}
\end{aligned}$$

then we obtain:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

This relation says that given the elongations of two elements perpendicular each other, then the change of angle between them is not arbitrary. Two similar equations are obtained in the same way for other directions.

2. By calculating the derivatives of shear strains:

$$2\,\frac{\partial \varepsilon_{12}}{\partial x_3} = \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3}$$

$$2\,\frac{\partial \varepsilon_{23}}{\partial x_1} = \frac{\partial^2 u_2}{\partial x_3 \partial x_1} + \frac{\partial^2 u_3}{\partial x_2 \partial x_1}$$

$$2\,\frac{\partial \varepsilon_{31}}{\partial x_2} = \frac{\partial^2 u_3}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_3 \partial x_2}$$

we obtain :

$$\frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = 2\,\frac{\partial^3 u_1}{\partial x_1 \partial x_2 \partial x_3} = 2\,\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3}$$

This relation says that when we know the angle changes of three elements that are perpendicular to each other, then the elongations can not be arbitrary. Similar procedure can be followed and finally we obtain the six strain compatibility equations:

$$\begin{aligned}
\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} &= 2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1\,\partial x_2} \\
\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} &= 2\,\frac{\partial^2 \varepsilon_{23}}{\partial x_2\,\partial x_3} \\
\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} &= 2\,\frac{\partial^2 \varepsilon_{31}}{\partial x_3\,\partial x_1} \\
\frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} \\
\frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} \\
\frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2}
\end{aligned} \tag{3.68}$$

The above equations can be written in the following form using the permutation symbol (2.7)

$$\epsilon_{phi}\,\epsilon_{mjk}\,\varepsilon_{ki,hj} = 0 \tag{3.69}$$

where we have used the notation for partial derivation (2.53):

$$\varepsilon_{ki,hj} = \frac{\partial^2 \varepsilon_{ki}}{\partial x_h\,\partial x_j}$$

or in the form

$$\mathcal{R}_{ijkl} = \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{il,jk} - \varepsilon_{jk,il} = 0 \tag{3.70}$$

The object $\mathcal{R}_{ijkl}$ generally has $3^4 = 81$ components, but only six non-zero components (the equation (3.68)) because of some symmetry (symmetry of the infinitesimal strain tensor, the partial derivations etc.).

3.10 Other strain tensors

Consider a cylindrical bar which has a uniform cross section S throughout its length l under uniaxial tension test (see figure (3.6)). Suppose the total load on the end of a bar is $\mathbf{P}$. The cross section area and the length before loading are respectively S_0 and l_0. To describe the change of the form of the bar, we can use a measure called *the stretch ratio* or *extension ratio*. It is defined as the ratio between the final length l and the initial length l_0:

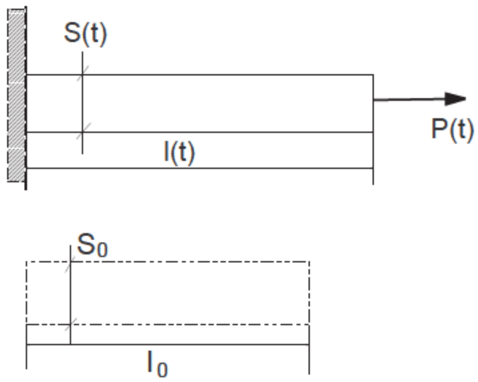

Figure 3.6: Uniaxial tension

$$\lambda = \frac{l}{l_0} \tag{3.71}$$

The stretch ratio is used in the analysis of materials that exhibit large deformations, such as rubber, elastomers, which can sustain stretch ratios of $\lambda = 2, 3, ...$ before they fail ([1]). On the other hand, traditional engineering materials, such as concrete or steel, fail at much lower stretch ratios $0,999 < \lambda < 1,001$. To avoid the problem of significant digits in calculation, we take another measure of strain as function of this stretch ratio:

$$\varepsilon = f(\lambda) \qquad f(1) = 0,\ f'(1) = 1 \tag{3.72}$$

where f is any function satisfying conditions $f(1) = 0$ and $f'(1) = 1$. The first condition $f(1) = 0$ is obvious, the second condition $f'(1) = 1$ assures that in case of infinitesimal strain, all these measures are equal to the engineering strain (3.75). Also for assuring one-to-one correspondence we take f monotonic, e.g. satisfying condition $f' > 0$ for $\lambda > 0$. The Taylor expansion of this function is:

$$\varepsilon = f(1) + (\lambda - 1)\frac{df}{d\lambda} + \frac{1}{2}(\lambda - 1)^2\frac{d^2 f}{d\lambda^2} + ... \tag{3.73}$$

For example, following function used by Hill satisfies all above conditions

$$\varepsilon(n) = f(\lambda) = \frac{\lambda^{2n} - 1}{2n} \tag{3.74}$$

where n can take any value. Figure (3.7) shows some curves for different value of n:

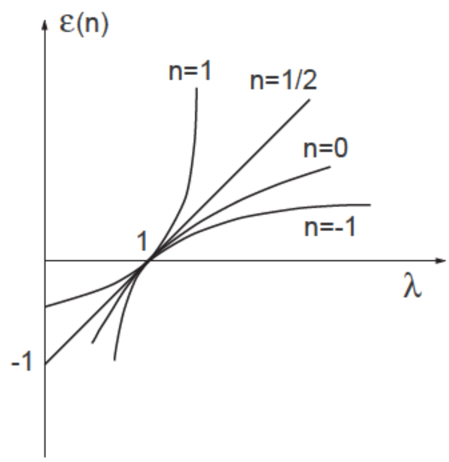

Figure 3.7:

The case when $n = 1/2$ gives us the Cauchy strain or engineering strain

$$\varepsilon = \frac{l - l_0}{l_0} \tag{3.75}$$

When $n = 1$ we have the Green measure of strain (3.17), $n = -1$ corresponds to the Euler strain tensor (3.22). When $n \to 0$

$$\varepsilon(0) = \lim_{n \to 0} \varepsilon(n) = \lim_{n \to 0} \frac{\lambda^{2n} - 1}{2n} = \ln \lambda \tag{3.76}$$

we obtain the logarithmic strain, also called true strain or Hencky strain.

In general three-dimensional case, the deformation gradient $\mathbf{F}$ (3.5) can be decomposed into two parts:

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} \tag{3.77}$$

where $\mathbf{U}$ is a symmetric tensor and $\mathbf{R}$ is a proper orthogonal tensor (with $\det \mathbf{R} = 1$). The decomposition (3.77) is unique and called *polar decomposition*. Tensor $\mathbf{U}$ is known as the stretch tensor and relates to the deformation gradient by:

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} \tag{3.78}$$

Tensor $\mathbf{U}$ is a symmetric tensor of second order, hence it has three principal directions $\mathbf{N}_i$. From (3.78) tensor $\mathbf{U}$ is positive definite, so its three principal values U_i are positive. In case of uniaxial tension, the principal value U_1 is exactly the stretch ratio λ (3.71). Once $\mathbf{U}$ is obtained we can calculate $\mathbf{R}$ by the relation $\mathbf{R} = \mathbf{F}\,\mathbf{U}^{-1}$ and the Hill's family of strain measure takes the form:

$$\mathbf{E}(n) = \mathbf{f}(\mathbf{U}) = \sum_{i=1}^{3} \frac{1}{2n} (\mathbf{U}^{2n} - 1)\, \mathbf{N}_i \otimes \mathbf{N}_i \tag{3.79}$$

Equations (3.70) are the representation of tensor $\mathbf{E}(n)$ on principal axis of the stretch tensor $\mathbf{U}$. The meaning of each $\mathbf{E}(n)$ for every value of n is the same as mentioned before.

Chapter 4

Stress

4.1 Introduction

Continuous distribution of material in a volume is defined by a function known as *density* ϱ

$$\varrho = \frac{dm}{dV} \tag{4.1}$$

Then the mass of body $\mathcal{M}$ is:

$$\mathcal{M} = \int dm = \int_V \varrho dV \tag{4.2}$$

When $\varrho = const$, the body is homogeneous, then $\mathcal{M} = \varrho V$.

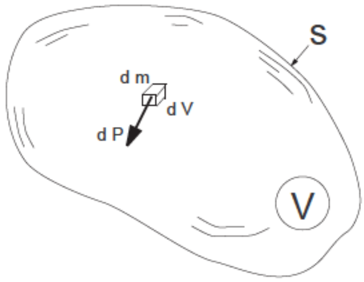

Figure 4.1: Body (mass) force

In the previous chapter we have studied the kinematics of bodies without considering of the forces causing the motion. There are two kinds of forces, namely body (mass) and surface forces, acting on bodies. The body (mass) forces are those that act on throughout a volume (mass) (for example: gravity, inertial forces, magnetic forces etc.). Surface forces are those that act on a surface, separating parts of the body.

Let the infinitesimal body force $d\mathbf{P}$ acts on the element of infinitesimal mass dm and of infinitesimal volume dV. The mass force is defined as:

$$\mathbf{b} = \frac{d\mathbf{P}}{dm} \tag{4.3}$$

and the body force is defined as a density of force and thus has units of newtons per cubic meter $[N/m^3]$:

$$\mathbf{b}^* = \frac{d\mathbf{P}}{dV} \tag{4.4}$$

By using (4.1) we have:

$$\mathbf{b}^* = \varrho\,\mathbf{b} \tag{4.5}$$

Problem 15 *Calculate the body (mass) force in gravity field.*

Solution: In gravity field $d\mathbf{P} = dm\,\mathbf{g}$. Here $\mathbf{g}$ is the gravity acceleration $|\mathbf{g}| = 9,81 m/s^2$, then the mass force is $\mathbf{b} = \mathbf{g}$ and the body force $\mathbf{b}^* = \varrho\,\mathbf{g}$.

4.2 Internal forces. Vector of stress

Consider a body shown in Figure (4.2). Imagine a surface such as S which passes through an arbitrary internal point M and has a normal unit vector $\mathbf{n}$. The surface divides the body into two parts. The interactions between these two parts have the character of surface forces and we call them internal forces. Considering the lower part as a free body, let $\Delta\mathbf{P}$ be a resultant force acting on a small area ΔS on surface S containing M. We define the stress vector (from upper part to lower part) at the point M as the limit:

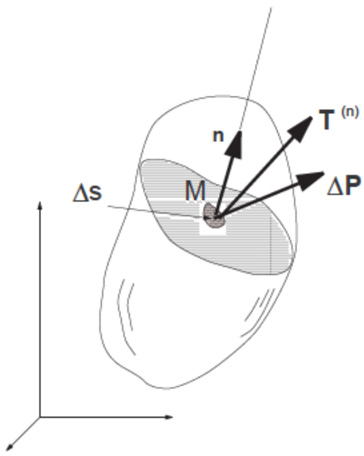

Figure 4.2: Vector of stress

$$\mathbf{T}^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta\mathbf{P}}{\Delta S} \tag{4.6}$$

Thus stress is defined as the intensity of force at a point. It has, as expected, units of newtons per square meter $[N/m^2]$. When upper portion is considered as a free body, following Newton'a law of action and reaction, we shall have the stress vector

$\mathbf{T}^{(-\mathbf{n})}$ at the same point on the same surface equal and opposite to that given by Eq (4.6), that is:

$$\mathbf{T}^{(-\mathbf{n})} = -\mathbf{T}^{(\mathbf{n})} \tag{4.7}$$

In general, vector of stress is non-collinear with the unit normal vector $\mathbf{n}$.

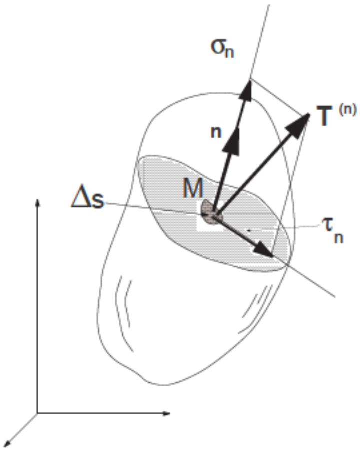

Figure 4.3: Normal stress and shear stress

$$\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma}_n + \boldsymbol{\tau}_n \tag{4.8}$$

It can be decomposed into $\boldsymbol{\sigma}_n$ called *normal stress* and $\boldsymbol{\tau}_n$ called *shear stress*. A normal stress (compressive or tensile) is one in which the force is normal to the area on which it acts. With a shear stress, the force is parallel to the area on which it acts. We have:

$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = T_i^{(\mathbf{n})} n_i \tag{4.9}$$

and for shear stress:

$$\tau_n = \sqrt{\mathbf{T}^{(\mathbf{n})} \cdot \mathbf{T}^{(\mathbf{n})} - (\sigma_n)^2} \tag{4.10}$$

4.3 Stress state. Cauchy's theorem. Stress tensor

We can imagine many such surfaces which pass through the point M and divide the body into two parts. For each surface, we have a unit normal vector $\mathbf{n}$ and the corresponding stress vector. Since there are an infinite number of cuts through the point M, we shall have an infinite number of vectors of stress, which are in general different from each other. When we know all these vectors, we say that we know the *stress state* at the point M.

Is it possible to know all these vectors of stress? The answer is *yes*, as we can see this fact in following theorem called Cauchy's theorem.

Let a small tetrahedron be isolated from the body with the point M as one of its vertices. The size of the tetrahedron will be made to approach zero volume

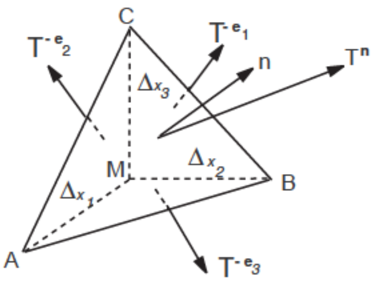

Figure 4.4:

in a way that in the limit, the inclined plane ABC will pass through the point M. The unit outward normal to the face MCB is $-\mathbf{e}_1$. The stress vector on this face is denoted by $\mathbf{T}^{-\mathbf{e}_1}$ and the force on it is $\mathbf{T}^{-\mathbf{e}_1}\,\Delta S_1$ where ΔS_1 is the area of MCB. Similarly, the forces acting on faces MAC, MAB and ABC are respectively $\mathbf{T}^{-\mathbf{e}_2}\,\Delta S_2$, $\mathbf{T}^{-\mathbf{e}_3}\,\Delta S_3$ and $\mathbf{T}^{\mathbf{n}}\,\Delta S$. Simple geometry reveals that:

$$\Delta S_1 = \frac{1}{2}\,\Delta x_2\,\Delta x_3;\; \Delta S_2 = \frac{1}{2}\,\Delta x_3\,\Delta x_1;\; \Delta S_3 = \frac{1}{2}\,\Delta x_1\,\Delta x_2 \tag{4.11}$$

Now we calculate the area of the oblique face ABC, and for this purpose we define two convenient vectors $\mathbf{AB}$, $\mathbf{AC}$ that define the oblique surface element. Using the properties of the vector product (see (2.19)) we have:

$$\Delta S\,\mathbf{n} = \frac{1}{2}\,\mathbf{AB}\times\mathbf{AC} \tag{4.12}$$

In terms of the intercepts and unit base vectors, this becomes

$$\Delta S\,\mathbf{n} = \frac{1}{2}\,(-\Delta x_1\mathbf{e}_1 + \Delta x_2\mathbf{e}_2)\times(-\Delta x_1\mathbf{e}_1 + \Delta x_3\mathbf{e}_3) = \tag{4.13}$$

$$= \frac{1}{2}\,(\Delta x_2\Delta x_3\,\mathbf{e}_1 + \Delta x_3\Delta x_1\,\mathbf{e}_2 + \Delta x_1\Delta x_2\,\mathbf{e}_3)$$

Thus, from (4.11) and (4.13)

$$\Delta S\,\mathbf{n} = \Delta S_i\,\mathbf{e}_i \;\Rightarrow\; \Delta S_i = \Delta S\,(\mathbf{n}\cdot\mathbf{e}_i) = \Delta S\,n_i \tag{4.14}$$

Writing the Newton's second law for the tetrahedron, now we have:

$$\sum \mathbf{F} = \mathbf{T}^{-\mathbf{e}_1}\,\Delta S_1 + \mathbf{T}^{-\mathbf{e}_2}\,\Delta S_2 + \mathbf{T}^{-\mathbf{e}_3}\,\Delta S_3 + \mathbf{T}^{\mathbf{n}}\Delta S + (\Delta m)\mathbf{b} = (\Delta m)\mathbf{a} \tag{4.15}$$

where the mass of the tetrahedron $\Delta m = (\mathit{density})(\mathit{volume})$, $\mathbf{b}$ is any mass force (see (4.3)) acting on the tetrahedron and $\mathbf{a}$ is the acceleration. Since the volume of the tetrahedron equals to $(1/6)\,\Delta x_1\,\Delta x_2\,\Delta x_3$ thus, it is proportional to the product of three infinitesimal lengths, then when the size of the tetrahedron approaches zero the acceleration and the mass force terms will approach zero faster than the first three terms in equation (4.15) where the stress vectors are multiplied by the areas (the product of two infinitesimal lengths), we obtain:

$$\mathbf{T}^{-\mathbf{e}_1}\,\Delta S_1 + \mathbf{T}^{-\mathbf{e}_2}\,\Delta S_2 + \mathbf{T}^{-\mathbf{e}_3}\,\Delta S_3 + \mathbf{T}^{\mathbf{n}}\Delta S = 0 \tag{4.16}$$

Using (4.15) $\Delta S_i = \Delta S\, n_i$, $(i = 1, 2, 3)$ we have:

$$\mathbf{T}^{-\mathbf{e}_1}\, n_1 + \mathbf{T}^{-\mathbf{e}_2}\, n_2 + \mathbf{T}^{-\mathbf{e}_3}\, n_3 + \mathbf{T}^{\mathbf{n}} = 0 \tag{4.17}$$

But from the law of action and reaction

$$\mathbf{T}^{-\mathbf{e}_1} = -\mathbf{T}^{\mathbf{e}_1};\ \mathbf{T}^{-\mathbf{e}_2} = -\mathbf{T}^{\mathbf{e}_2};\ \mathbf{T}^{-\mathbf{e}_3} = -\mathbf{T}^{\mathbf{e}_3}$$

then equations (4.17) becomes:

$$\mathbf{T}^{\mathbf{n}} = \mathbf{T}^{\mathbf{e}_1}\, n_1 + \mathbf{T}^{\mathbf{e}_2}\, n_2 + \mathbf{T}^{\mathbf{e}_3}\, n_3 = \mathbf{T}^{\mathbf{e}_i}\, n_i \tag{4.18}$$

Equation (4.18) represents the Cauchy's theorem. It says that when we know three vectors of stress $\mathbf{T}^{\mathbf{e}_1}, \mathbf{T}^{\mathbf{e}_2}, \mathbf{T}^{\mathbf{e}_3}$ for the three mutually perpendicular area elements whose normals are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, then we know the vector of stress for any plane with unit outward vector $\mathbf{n}$, and thus we knows the stress state in the point M.

Decompose the stress vectors $\mathbf{T}^{\mathbf{e}_1}, \mathbf{T}^{\mathbf{e}_2}, \mathbf{T}^{\mathbf{e}_3}$ on axes:

$$\begin{aligned} \mathbf{T}^{\mathbf{e}_1} &= \sigma_{11}\,\mathbf{e}_1 + \sigma_{12}\,\mathbf{e}_2 + \sigma_{13}\,\mathbf{e}_3 \\ \mathbf{T}^{\mathbf{e}_2} &= \sigma_{21}\,\mathbf{e}_1 + \sigma_{22}\,\mathbf{e}_2 + \sigma_{23}\,\mathbf{e}_3 \\ \mathbf{T}^{\mathbf{e}_3} &= \sigma_{31}\,\mathbf{e}_1 + \sigma_{32}\,\mathbf{e}_2 + \sigma_{33}\,\mathbf{e}_3 \end{aligned} \tag{4.19}$$

Since $\mathbf{T}^{\mathbf{e}_1}$ is the vector of stress acting on the plane whose outward normal is $\mathbf{e}_1$, then σ_{11} is its normal component and σ_{12}, σ_{13} are its shear components. In the same way we have the normal and shear component on others planes. This suggests the definition of a stress tensor $\boldsymbol{\sigma}$:

$$\mathbf{T}^{\mathbf{n}} = \overset{T}{\boldsymbol{\sigma}}\,\mathbf{n} \;\Rightarrow\; T_i^{\mathbf{n}} = \sigma_{ji} n_j \tag{4.20}$$

and the matrix representation of stress tensor $\boldsymbol{\sigma}$ in the bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{21} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \tag{4.21}$$

Note that for each stress components σ_{ij} the first index i indicates the plane on which the stress component acts, and the second index indicates the direction of the component. For example σ_{12} is the stress component acting on the direction $\mathbf{e}_2$ on the plane whose outward normal is $\mathbf{e}_1$. Both σ_{12} and σ_{13} are shearing stresses acting

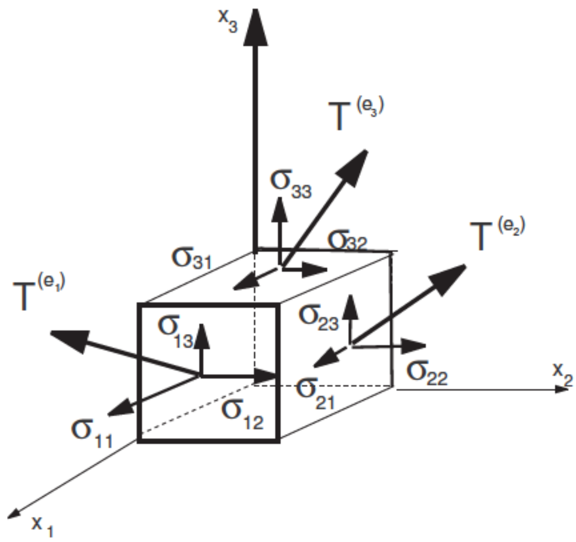

Figure 4.5: Stress tensor

on the same plane with outward normal $\mathbf{e}_1$, thus the resultant shearing stress on this plane is given by:

$$\boldsymbol{\tau}_1 = \sigma_{12}\,\mathbf{e}_2 + \sigma_{13}\,\mathbf{e}_3$$

and the magnitude of this is $\sqrt{\sigma_{12}^2 + \sigma_{13}^2}$. Note also that some authors use the other convention: first index indicates the direction and the second index indicates the plane, but these differences in meaning regarding the shear components disappear because the stress tensor is symmetric as we can show later using the principle of moment of momentum (see (5.20)). Equation (4.20) can be rewritten somewhat more conventionally as

$$\mathbf{T}^{\mathbf{n}} = \boldsymbol{\sigma}\,\mathbf{n} \;\Rightarrow\; T_i^{\mathbf{n}} = \sigma_{ij} n_j \tag{4.22}$$

At this place we can see the crucial difference between force and stress. As shown in figure (4.6) when we decompose the force $\mathbf{F}$ then $F_n = F\cos\alpha$, but if we calculate for example the normal stress, then $\sigma_n = \sigma\cos^2\alpha$ because both the force and area are revolved in this second case. This is the key for understanding stress components. Remember also that in figure (4.5) we use the familiar symbol of vector to denote the stresses, but they are not vectors and can not be added by the rule of a parallelogram as in the case of forces.

4.4 Principal stresses

Since stress tensor is a symmetric tensor of second order (see Section (2.10)), there exist at least three mutually perpendicular principal directions. The planes having these directions as their normals are called *principal directions*, on these planes the stress vectors are normal to them (i.e. no shearing stresses). These normal stresses are known as *principal stresses*. Principal stresses and principal

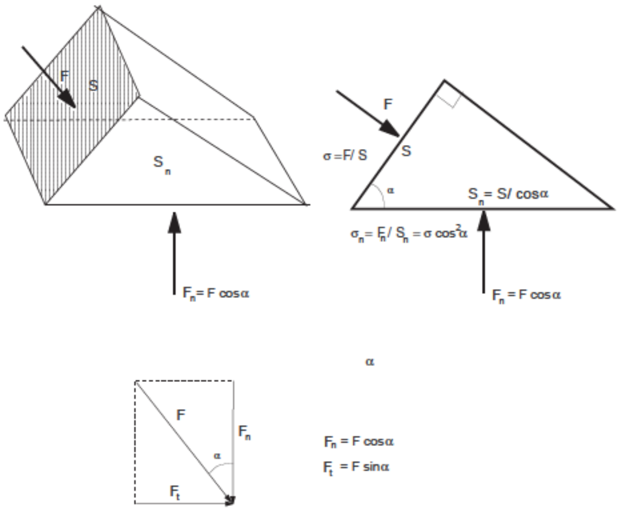

Figure 4.6:

directions can be found from Equations (2.57)

$$(\sigma_{ik} - \sigma\delta_{ik})\, n_k = 0 \;\text{ with }\; n_i n_i = 1 \tag{4.23}$$

From (4.23) we can also calculate the principal stresses by the characteristic equation of tensor $\boldsymbol{\sigma}$ (see (2.59)):

$$\det|\sigma_{ik} - \sigma\delta_{ik}| = 0 \tag{4.24}$$

or

$$\sigma^3 - I_\sigma \sigma^2 + II_\sigma \sigma - III_\sigma = 0 \tag{4.25}$$

where

$$\begin{aligned} I_\sigma &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\ II_\sigma &= \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{33} & \sigma_{31} \\ \sigma_{13} & \sigma_{11} \end{vmatrix} \\ III_\sigma &= \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \end{aligned} \tag{4.26}$$

are the principal scalar invariants of the stress tensor. The equation (4.25) has three real roots, denoted by $\sigma_1, \sigma_2, \sigma_3$. They are principal stresses.

Problem 16 *The stress tensor at a certain point of a body is given by [2]:*

$$\begin{pmatrix} 7 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 4 \end{pmatrix} (MPa) \tag{4.27}$$

1. Find the stress vector, its magnitude, magnitude of normal stress, shear stress and the angle between this vector on the plane passing through the point having the unit normal $\mathbf{n} = (2/3)\mathbf{e}_1 + (-2/3)\mathbf{e}_2 + (-1/3)\mathbf{e}_3$.

2. Find the principal stresses and principal directions.

3. If

$$\mathbf{e}'_1 = \frac{1}{3}\,(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$$

and

$$\mathbf{e}'_2 = \frac{1}{\sqrt{2}}\,(\mathbf{e}_1 - \mathbf{e}_2)$$

find σ'_{12}.

Solution:

1. The stress vector is obtained from the equation (4.20) as:

$$(\mathbf{T}^{\mathbf{n}}) = \begin{pmatrix} 7 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 4 \\ -10/3 \\ 0 \end{pmatrix}$$

or

$$\mathbf{T}^{\mathbf{n}} = 4\mathbf{e}_1 - (10/3)\mathbf{e}_2$$

The magnitude of this vector is

$$|\mathbf{T}^{\mathbf{n}}| = \sqrt{(4)^2 + (-10/3)^2} = \sqrt{244}/3\,MPa$$

The magnitude of the normal stress simply is:

$$|\boldsymbol{\sigma}_n| = \mathbf{T}\cdot\mathbf{n} = (4)(2/3) + (-10/3)(-2/3) + (0)(-1/3) = 44/9\,MPa$$

The magnitude of the shear stress is:

$$|\boldsymbol{\tau}_n| = \sqrt{|\mathbf{T}^{\mathbf{n}}|^2 - |\boldsymbol{\sigma}_n|^2} = \sqrt{(244/9 - (44/9)^2} = \sqrt{260}/9\,MPa$$

The cosine of angle ϑ between vectors $\mathbf{T}^{\mathbf{n}}$ and $\mathbf{n}$:

$$\cos\vartheta = \frac{|\boldsymbol{\sigma}_n|}{|\mathbf{T}^{\mathbf{n}}|} \approx 0.94\,;\ \vartheta \approx 20^0$$

2. From Eqs (4.24)

$$\begin{vmatrix} (7-\sigma) & 0 & 2 \\ 0 & (5-\sigma) & 0 \\ 2 & 0 & (4-\sigma) \end{vmatrix} = (7-\sigma)(5-\sigma)(4-\sigma) - 4(5-\sigma) = 0$$

$$(5-\sigma)[(7-\sigma)(4-\sigma) - 4] = 0$$

we find the principal stresses $\sigma_1 = 8, \sigma_2 = 5, \sigma_3 = 3\ (MPa)$.

For $\sigma_1 = 8$ we have from (4.23):

$$\begin{cases} (7-8)n_1 + 0 \cdot n_2 + 2n_3 &= 0 \\ 0 \cdot n_1 + (5-8)n_2 + 0 \cdot n_3 &= 0 \\ 2n_1 + 0 \cdot n_2 + (4-8)n_3 &= 0 \\ (n_1)^2 + (n_2)^2 + (n_3)^2 &= 1 \end{cases}$$

From the second equation we have $n_2 = 0$, the first and third equation gives $n_1 = 2n_3$. Putting these values to the forth equation we find $n_3 = \pm 1/\sqrt{5}$, then we have the first principal direction $\mathbf{n}^{(1)} = (\pm 2/\sqrt{5})\mathbf{e}_1 + (\pm 1/\sqrt{5})\mathbf{e}_3$. For $\sigma_2 = 5$:

$$\begin{cases} (7-5)n_1 + 0 \cdot n_2 + 2n_3 &= 0 \\ 0 \cdot n_1 + (5-5)n_2 + 0 \cdot n_3 &= 0 \\ 2n_1 + 0 \cdot n_2 + (4-5)n_3 &= 0 \\ (n_1)^2 + (n_2)^2 + (n_3)^2 &= 1 \end{cases}$$

$$\begin{cases} 2n_1 + 2n_3 &= 0 \\ 0 &= 0 \\ 2n_1 - n_3 &= 0 \\ (n_1)^2 + (n_2)^2 + (n_3)^2 &= 1 \end{cases}$$

then $n_1 = n_3 = 0$, $n_2 = \pm 1$ and $\mathbf{n}^{(2)} = (\pm 1)\mathbf{e}_2$. Similarly for $\sigma_3 = 3$ we find $\mathbf{n}^{(3)} = (\pm 1/\sqrt{5})\mathbf{e}_1 + (\mp 2/\sqrt{5})\mathbf{e}_3$.

The three principal directions are mutually perpendicular, we can check this by calculating the scalar product $\mathbf{n}^{(i)} \cdot \mathbf{n}^{(j)} = 0$ for $i \neq j$.

Taking for example three vectors $\mathbf{n}^{(1)} = (2/\sqrt{5})\mathbf{e}_1 + (1/\sqrt{5})\mathbf{e}_3$, $\mathbf{n}^{(2)} = 1$, $\mathbf{n}^{(3)} = (-1/\sqrt{5})\mathbf{e}_1 + (2/\sqrt{5})\mathbf{e}_3$. They form a right-handed system of coordinate because the triple product of them is positive (equals to 1) (see (2.21)). In this coordinate system, the stress tensor has a simple form:

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where six components are equal to zero.

3. To find the primed component we use (2.50) $T'_{12} = \mathbf{e}'_1 \cdot (\mathbf{T}\,\mathbf{e}'_2)$:

$$T'_{12} = (2/3,\ \ 2/3,\ \ 1) \begin{pmatrix} 7 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} = (2/3,\ 2/3,\ 1) \begin{pmatrix} 7/\sqrt{2} \\ -5/\sqrt{2} \\ 7/\sqrt{2} \end{pmatrix} =$$

$$= \frac{25}{3\sqrt{2}}\,MPa$$

4.5 Decomposition of the stress tensor

Stress tensor can be decomposed into two parts as in Section (2.11). Denote by p the mean normal stress (or *hydrostatic pressure*):

$$p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\, I_\sigma \tag{4.28}$$

Tensor $p\delta_{ij}$ represents *hydrostatic tension*:

$$\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \tag{4.29}$$

then the *deviatoric stress tensor* is defined by

$$s_{ij} = \sigma_{ij} - p\,\delta_{ij} = \sigma_{ij} - \frac{1}{3}\, I_\sigma\, \delta_{ij} \tag{4.30}$$

or

$$s_{ij} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{pmatrix} \tag{4.31}$$

Subtracting a constant normal stress in all directions will not change the principal directions, so the principal directions are the same for the deviatoric stress tensor as for the original stress tensor. We can also find them from the equations:

$$(s_{ij} - s\,\delta_{ij})\, n_j = 0 \tag{4.32}$$

here the characteristic equation has the form:

$$s^3 - II_s\, s - III_s = 0 \tag{4.33}$$

where the invariants of the deviatoric stress tensor are:

$$\begin{aligned} I_s &= s_{ii} \\ II_s &= \frac{1}{2}\, s_{ij}\, s_{ji} \\ III_s &= \frac{1}{3}\, s_{ij}\, s_{jk}\, s_{ki} = \det(s_{ij}) \end{aligned} \tag{4.34}$$

A state which the mean normal stress $s_{ii} = 0$ is called *pure shear*. Denoting by s_1, s_2 and s_3 the principal values of deviatoric stress tensor, we also have:

$$\begin{aligned} I_s &= s_1 + s_2 + s_3 = 0 \\ II_s &= (s_1 s_2 + s_2 s_3 + s_3 s_1) \\ III_s &= s_1 s_2 s_3 \end{aligned} \tag{4.35}$$

Also the invariants of the deviatoric stress tensor can be calculated in terms of principal values of the stress tensor $\sigma_1, \sigma_2, \sigma_3$:

$$\begin{aligned} I_s &= 0 \\ II_s = \frac{1}{3}\left(I_\sigma^2 - 3\,II_\sigma\right) &= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] \\ III_s &= \frac{1}{27}\left(2\,I_\sigma^3 - 9\,I_\sigma\,II_\sigma + 27\,III_\sigma\right) \end{aligned} \tag{4.36}$$

Thus, any stress state σ_{ij} can be decomposed into two stress state, one of which is pure shear s_{ij} and the other is hydrostatic tension $p\,\delta_{ij}$. The deviatoric stress tensor plays a very important role in the stress-strain relationship discussed later in this course.

4.6 Principal shear stresses

Suppose that x_1, x_2, x_3 are principal axes. The normal to a plane is n_i and the normal stress σ_n on this plane given by (4.22):

$$\sigma_n = \sigma_{ij}\,n_i\,n_j = \sigma_1\,n_1^2 + \sigma_2\,n_2^2 + \sigma_3\,n_3^2 \tag{4.37}$$

where σ_1, σ_2, σ_3 are principal stresses. Then the shear stress on this plane τ_n is given by (4.10)

$$\begin{aligned} (\tau_n)^2 = (\sigma_1\,n_1)^2 + (\sigma_2\,n_2)^2 + (\sigma_3\,n_3)^2 - \sigma_n^2 = (\sigma_1\,n_1)^2 + (\sigma_2\,n_2)^2 + (\sigma_3\,n_3)^2 - \\ -(\sigma_1\,n_1^2 + \sigma_2\,n_2^2 + \sigma_3\,n_3^2) \end{aligned} \tag{4.38}$$

For known values of σ_1, σ_2, σ_3, equation (4.38) is an equation of n_i. Calculating the extremum of this function, it follows that the maximum shear stress is one half the largest difference between any two of the principal stresses and occurs on an area element whose unit normal makes an angle of 45^0 with each of the corresponding principal axes. The quantities

$$\tau_1 = \frac{1}{2}\,|\sigma_1 - \sigma_3|,\ \ \tau_2 = \frac{1}{2}\ |\sigma_3 - \sigma_1|,\ \ \tau_3 = \frac{1}{2}\,|\sigma_1 - \sigma_2| \tag{4.39}$$

are called *principal shear stresses*. The largest numerical value of the principal shears is called the *maximum shear stress*. For $\sigma_1 > \sigma_2 > \sigma_3$, then $\tau_{max} = (1/2)\,|\sigma_1 - \sigma_3|$ or $\tau_{max} = \max(\tau_1,\ \tau_2,\ \tau_3)$.

4.7 Other stress tensors

The stress tensor discussed above is called the Cauchy's stress tensor. It is defined as the ratio of actual force on the deformed area.

Consider now the body at two configurations: a reference (undeformed) and an actual (deformed) configuration under the transformation (3.1). Let the density in the reference configuration be ϱ_0, the body force - $\mathbf{b}_0$, the undeformed area - dS. In the actual configuration we denote the density by ϱ, the body force by $\mathbf{b}$ and the deformed area by ds. Let $\mathbf{N}$ be the unit normal of dS, while $\mathbf{n}$ is the unit normal of ds. We can show that

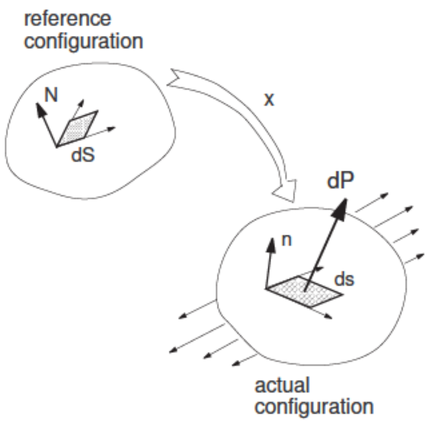

Figure 4.7:

$$\begin{aligned} J &= \det\mathbf{F} \\ \mathbf{b}_0 &= J\,\mathbf{b} \\ \varrho_0 &= J\,\varrho \\ \mathbf{n}\,ds &= J\,\mathbf{N}\,\mathbf{F}^{-T}\,dS \end{aligned} \tag{4.40}$$

where J is the Jacobian (3.2), $\mathbf{F}$ is deformation gradient in (3.5). We have different definitions of stress:

$$\boldsymbol{\sigma}\,\mathbf{n} = \lim_{ds\to 0}\frac{d\mathbf{P}}{ds} \tag{4.41}$$

$\Rightarrow$ the Cauchy's stress tensor $\boldsymbol{\sigma}$

$$\mathbf{S}\,\mathbf{N} = \lim_{dS\to 0}\frac{d\mathbf{P}}{dS} \tag{4.42}$$

$\Rightarrow$ the first Piola-Kirchoff stress tensor $\mathbf{S} = J\,\boldsymbol{\sigma}\,\mathbf{F}^{-T}$

$$\boldsymbol{\pi}\,\mathbf{N} = \lim_{dS\to 0}\frac{\mathbf{F}^{-1}d\mathbf{P}}{dS} \tag{4.43}$$

$\Rightarrow$ the second Piola-Kirchoff stress tensor $\boldsymbol{\pi} = J\,\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$

Physical interpretation of different stress tensors

* Components of the Cauchy's stress tensor $\boldsymbol{\sigma}$ are densities of actual force acting on deformed area.

* Components of the first Piola-Kirchhoff stress tensor (nominal stress tensor) $\mathbf{S}$ are densities of actual force acting on undeformed area.

* There are no simple interpretations for the components of the second Piola-Kirchhoff stress tensor.

These stress tensors are the generalized forces conjugated with generalized measures of deformation $\mathbf{D}$, $\mathbf{F}$ and $\mathbf{E}$, because of:

$$\boldsymbol{\sigma} \cdot \mathbf{D} = \mathbf{S} \cdot \dot{\mathbf{F}} = \boldsymbol{\pi} \cdot \dot{\mathbf{E}} \tag{4.44}$$

In case of infinitesimal strain: $\mathbf{F} = (\mathbf{1} + \text{grad}\mathbf{u}),\ \det\mathbf{F} \approx (1 + \text{div}\mathbf{u})$ then

$$\boldsymbol{\sigma} \approx \mathbf{S} \approx \boldsymbol{\pi} \tag{4.45}$$

Chapter 5

Conservation Laws

5.1 Introduction

We know already how to calculate the material derivative (time rate change) of a quantity of a material particle presented in section (3.5). Now we want to calculate the time rate change dI/dt of a physical quantity (like density, energy, momentum...) relating to a material volume element V in motion. At an instant of time t, a body occupies a regular region V of space with boundary S. Examine now integral of the type:

$$I = \int_{V(t)} M(\mathbf{x}, t)\, dV \tag{5.1}$$

where $M(\mathbf{x}, t)$ is a function of spatial coordinate $\mathbf{x}$ and time t. The volume V is also a function of t. Since this integral is taken over a fixed amount of mass (i.e.,

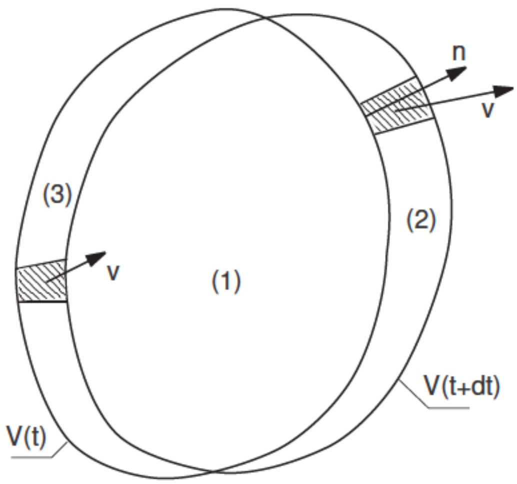

Figure 5.1:

material), we may interchange the order of differentiation and integration. This leads to:

$$\frac{dI}{dt} = \frac{d}{dt}\int\limits_{V(t)} M(\mathbf{x},t)dV = \int\limits_{V(t)} \frac{d}{dt}M(\mathbf{x},t)dV = \int\limits_{V(t)} \frac{dM(\mathbf{x},t)}{dt}dV + \int\limits_{V(t)} M\,\frac{d}{dt}(dV) \tag{5.2}$$

The material derivative of M similarly as in (3.33), (3.34), and the rate of change of a volume element integral can be shown basing on (3.64):

$$\frac{d}{dt}\,dV = (\mathrm{div}\mathbf{v})\,dV$$

then

$$\frac{dI}{dt} = \int\limits_{V}\left[\left(\frac{\partial M}{\partial t} + \mathrm{grad}\,M\cdot\mathbf{v}\right) + M\mathrm{div}\mathbf{v}\right]\,dV \tag{5.3}$$

Denoting the material derivative of M by

$$\dot{M} = \frac{\partial M}{\partial t} + \mathrm{grad}\,M\cdot\mathbf{v} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x_i}v_i \tag{5.4}$$

we obtain:

$$\frac{d}{dt}\int\limits_{V(t)} M(\mathbf{x},t)\,dV = \int\limits_{V}\left(\dot{M} + M\,\mathrm{div}\mathbf{v}\right)\,dV \tag{5.5}$$

Also, using the Gauss (divergence) theorem (2.69) for the last two terms in (5.4) to obtain another form of the rate, showing the contribution of particles in regions (1), (2), (3) on figure 5.1:

$$\frac{d}{dt}\int\limits_{V(t)} M(\mathbf{x},t)\,dV = \int\limits_{V}\frac{\partial M}{\partial t}\,dV + \int\limits_{S} M\,(\mathbf{v}\cdot\mathbf{n})\,dS \tag{5.6}$$

All global conservation laws have the following form:

$$\frac{d}{dt}\int\limits_{V(t)} M(\mathbf{x},t)\,dV = \int\limits_{V} f dV + \int\limits_{S} g dS \tag{5.7}$$

where functions M, f are defined on V, while g is defined on S.

5.2 Conservation of mass. Continuity equation

If we follow a volume V of material through its motion, its volume and density may change, but its total mass will remain unchanged. Taking $M = \varrho$ in (5.6), the integral $\int\limits_V \varrho dV$ is now the mass of the body. In this case in relation (5.7) functions $f = g = 0$. Because of the mass conservation, we have:

$$\frac{d}{dt}\int_V \varrho dV = 0 \tag{5.8}$$

Since this is valid for every region V, then locally we have

$$\dot{\varrho} + \varrho\,\mathrm{div}\mathbf{v} = 0 \tag{5.9}$$

This equation is the equation of conversation of mass, also called equation of continuity. In Cartesian coordinate system, the equations read:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x_1} v_1 + \frac{\partial \varrho}{\partial x_2} v_2 + \frac{\partial \varrho}{\partial x_3} v_3 + \varrho \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) = 0 \tag{5.10}$$

For an incompressible material, the material derivative of the density is zero, then the equation of continuity reads:

$$\mathrm{div}\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \tag{5.11}$$

In problems of statics, the equation of continuity is identically satisfied.

5.3 Conservation of momentum. Equation of motion

Each particle of the body must satisfy the Newton's equation of motion, and for the whole body, we must have the conservation of momentum. Let function M in (5.6) be the momentum: $M = \varrho\,\mathbf{v}$ where $\mathbf{v}$ is the particle velocity. The left-hand side of (5.6) now is the rate of momentum, then the right-hand side must be sum of forces acting on the body, namely body and surface forces, hence $f = \varrho\,\mathbf{b}$, $g = \mathbf{T}^{(n)}$. The global conversation law of momentum reads:

$$\frac{d}{dt} \int_V \varrho \mathbf{v} dV = \int_V \varrho\,\mathbf{b} dV + \int_S \mathbf{T}^{(n)} dS \tag{5.12}$$

where S is the bounding surface of the region in question. Applying (5.3) for material derivative and using (4.22) for $\mathbf{T}^{(n)}$, we obtain:

$$\int_V \left[\frac{d(\varrho \mathbf{v})}{dt} + (\varrho\,\mathbf{v})\mathrm{div}\,\mathbf{v} \right] dV = \int_V \varrho\,\mathbf{b} dV + \int_S (\boldsymbol{\sigma}\mathbf{n}) dS \tag{5.13}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{n}$ is the outward pointing unit normal to S. Use the Gauss theorem (2.69) for the last term:

$$\int_V [\,\dot{\varrho}\mathbf{v} + \varrho\,\dot{\mathbf{v}} + (\varrho\,\mathbf{v})\,\mathrm{div}\,(\mathbf{v}) - \varrho\,\mathbf{b} - \mathrm{div}\,\boldsymbol{\sigma}]\, dV = 0 \tag{5.14}$$

then

$$\int_V \left[\underbrace{(\dot{\varrho} + \varrho\,\mathrm{div}\,\mathbf{v}\,)}_{0} \mathbf{v} + \varrho\,\dot{\mathbf{v}} - \varrho\,\mathbf{b} - \mathrm{div}\,\boldsymbol{\sigma} \right] dV = 0$$

Following the equation of continuity (5.9): $\dot{\varrho}+\varrho\,\mathrm{div}\,\mathbf{v}=0$ we can rewrite the equation in the form:

$$\int\limits_V (\mathrm{div}\,\boldsymbol{\sigma} + \varrho\,\mathbf{b} - \varrho\,\dot{\mathbf{v}})dV = 0 \tag{5.15}$$

Because V is arbitrary, we obtain from (5.15) the local form for momentum conversation:

$$\mathrm{div}\boldsymbol{\sigma} + \varrho\,\mathbf{b} = \varrho\,\dot{\mathbf{v}} \tag{5.16}$$

This equation is called *the equation of motion.* If the acceleration vanishes, then equation (5.16) reduces to *the equilibrium equation*:

$$\mathrm{div}\boldsymbol{\sigma} + \varrho\,\mathbf{b} = \mathbf{0} \tag{5.17}$$

In Cartesian coordinate system, the equation (5.16) reads:

$$\begin{cases} \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} + \dfrac{\partial\sigma_{13}}{\partial x_3} + \varrho b_1 &= \varrho\dot{v}_1 \\[2ex] \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} + \dfrac{\partial\sigma_{23}}{\partial x_3} + \varrho b_2 &= \varrho\dot{v}_2 \\[2ex] \dfrac{\partial\sigma_{31}}{\partial x_1} + \dfrac{\partial\sigma_{32}}{\partial x_2} + \dfrac{\partial\sigma_{33}}{\partial x_3} + \varrho b_3 &= \varrho\dot{v}_3 \end{cases} \tag{5.18}$$

5.4 Conservation of moment of momentum. Symmetry of Cauchy's stress tensor

Let now $M = \mathbf{r}\times\varrho\,\mathbf{v}$ be the moment of momentum about a reference origin of the Cartesian coordinate O, where we denote by $\mathbf{r}$ the radius vector that represents the position of a point in space. Now $f = \mathbf{r}\times\varrho\,\mathbf{b}$, $g = \mathbf{r}\times\mathbf{T}^{(n)}$ are moment of body and surface force about origin O. The global law of conversation of moment of momentum reads:

$$\frac{d}{dt}\int\limits_V \mathbf{r}\times\varrho\mathbf{v}\,dV = \int\limits_V \mathbf{r}\times\varrho\,\mathbf{b}\,dV + \int\limits_S \mathbf{r}\times\mathbf{T}^{(n)}\,dS \tag{5.19}$$

Introducing Cauchy's formula $\mathbf{T}^{(n)} = \boldsymbol{\sigma}\,\mathbf{n}$ into the last integral, we have:

$$\frac{d}{dt}\int\limits_V \mathbf{r}\times\varrho\mathbf{v}\,dV = \int\limits_V \mathbf{r}\times\varrho\,\mathbf{b}\,dV + \int\limits_S \mathbf{r}\times(\boldsymbol{\sigma}\,\mathbf{n})\,dS$$

Using (2.20) we write this expression in the index form:

$$\frac{d}{dt}\int\limits_V \epsilon_{ijk}x_j\varrho\,v_k\,dV = \int\limits_V \epsilon_{ijk}x_j\varrho\,b_k\,dV + \int\limits_S \epsilon_{ijk}\,x_j(\sigma_{kp}\,n_p)\,dS$$

Transforming the last integral into a volume integral by Gauss's theorem:

$$\frac{d}{dt}\int_V \epsilon_{ijk}\,x_j\,\varrho\,v_k dV = \int_V \epsilon_{ijk}\,x_j\,\varrho\,b_k\,dV + \int_V \epsilon_{ijk}\left[x_{j,p}(\sigma_{kp}) + x_j\,\sigma_{kp,p}\right]dV$$

Following the same procedure as for the momentum: evaluating the material derivative according to (5.5), because V is arbitrary then the expression under the integral must be zero:

$$\epsilon_{ijk}\,\dot{x}_j\;\varrho\,v_k+\epsilon_{ijk}\,x_j\,\dot{\varrho}\,v_k+\epsilon_{ijk}\,x_j\,\varrho\,\dot{v}_k+\epsilon_{ijk}\,x_j\,\varrho\,v_k\,v_{m,m} = \epsilon_{ijk}\,x_j\,\varrho\,b_k+\epsilon_{ijk}\,\delta_{jp}\,\sigma_{kp}+\epsilon_{ijk}\,x_j\,\sigma_{kp,p}$$

The first term equals to zero, because this is the vector product of vectors $\mathbf{v}$ and $\varrho\,\mathbf{v}$; $\delta_{jp}\,\sigma_{kp} = \sigma_{kj}$. Hence this equation becomes:

$$-\epsilon_{ijk}\,x_j\,\dot{\varrho}\,v_k - \epsilon_{ijk}\,x_j\,\varrho\,\dot{v}_k - \epsilon_{ijk}\,x_j\,\varrho\,v_k\,v_{m,m} + \epsilon_{ijk}\,x_j\,\varrho\,b_k + \epsilon_{ijk}\,\sigma_{kj} + \epsilon_{ijk}\,x_j\,\sigma_{kp,p} = 0$$

or

$$\epsilon_{ijk}\,x_j\left(\sigma_{kp,p} + \varrho\,b_k - \varrho\,\dot{v}_k\right) - \epsilon_{ijk}\,x_j\,v_k\left(\dot{\varrho} + \varrho\,v_{m,m}\right) + \epsilon_{ijk}\,\sigma_{kj} = 0$$

The terms in brackets vanish by the equation of motion (5.16) and equation of continuity (5.9), this equation is reduced to

$$\varepsilon_{ijk}\sigma_{kj} = 0$$

i.e.

$$\sigma_{jk} = \sigma_{kj} \tag{5.20}$$

5.5 Conservation of energy. The first law of Thermodynamics

In continuum mechanics, a deforming body is considered as a thermodynamic system. The motion of a body must be governed by the law of conservation of energy (the first law of thermodynamics). This law relates the mechanical work done on the system and the heat transferred into the system to the change in total energy of the system. Let u be the specific internal energy (per unit mass), then the function $M = \varrho(u + 1/2\mathbf{v}\cdot\mathbf{v})$ in equation (5.7) now is the sum of internal and kinetic energy of the volume element, $f = \varrho\,\mathbf{b}\cdot\mathbf{v}$ is the rate of work of body forces, and $g = g_1 + g_2$. The function $g_1 = \mathbf{T}^{(n)}\cdot\mathbf{v}$ is the rate of external surface forces done on body while $g_2 = -\mathbf{q}\cdot\mathbf{n}$ gives the rate of heat flow by conduction across the surface S. Here $\mathbf{q}$ is a vector whose magnitude gives the rate of heat flow by conduction across a unit area. Vector $\mathbf{q}$ is called *the heat flux*, measured by $[q] = [J/m^2\,s]$.

First law of thermodynamics states that in any process the total energy of the system is conserved. Total change of the sum of internal dU/dt and kinetic energies

dC/dt of a system is equal to the the rate of work $d'W/dt$ done on and the rate of heat $d'Q/dt$ transferred into a system (see figure 5.2). We use the "prime" because $d'W$ and $d'Q$ are inexact, while dU and dC are exact differentials. The global form of the first law reads:

$$\underbrace{\frac{d}{dt}\int\limits_V \varrho\left(u+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)dV}_{\dfrac{dE}{dt}+\dfrac{dC}{dt}} = \underbrace{\int\limits_V \varrho\,\mathbf{b}\cdot\mathbf{v}dV + \int\limits_S \mathbf{T}^{(n)}\cdot\mathbf{v}dS}_{\dfrac{d'W}{dt}} \underbrace{-\int\limits_S \mathbf{q}\cdot\mathbf{n}\,dS}_{+\dfrac{d'Q}{dt}} \tag{5.21}$$

The minus sign in the last term is because $\mathbf{n}$ is unit outer normal to S therefore

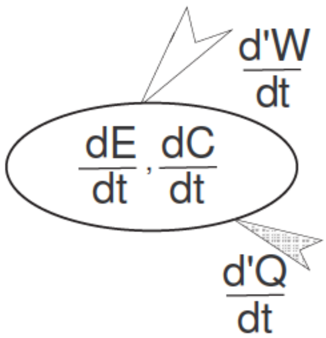

Figure 5.2:

$-\mathbf{q}\cdot\mathbf{n}$ represents inflow. Introducing Cauchy's formula $T_i^n = \sigma_{ij}\,n_j$ and writing in (5.21) index form:

$$\frac{d}{dt}\int\limits_V \varrho\left(u+\frac{1}{2}v_i\,v_i\right)dV = \int\limits_V \varrho\,b_i\,v_i\,dV + \int\limits_S \sigma_{ij}\,n_j\,v_i\,dS - \int\limits_S q_i\,n_i\,dS \tag{5.22}$$

Transforming the last two integrals into a volume integral by Gauss's theorem:

$$\frac{d}{dt}\int\limits_V \varrho\left(u+\frac{1}{2}v_i\,v_i\right)dV = \int\limits_V \varrho\,b_i\,v_i\,dV + \left(\int\limits_V \sigma_{ij,j}\,v_i\,dV + \int\limits_V \sigma_{ij}\,v_{i,j}\,dV\right) - \int\limits_V q_{i,i}\,dV \tag{5.23}$$

Following the same procedure as for the momentum: evaluating the material derivative according to (5.5), because V is arbitrary then the expression under the integral must be zero:

$$\varrho\dot{u} + \dot{\varrho}u + \varrho\,v_i\,\dot{v}_i + \frac{1}{2}\dot{\varrho}\,v_i\,v_i + \varrho(u+\frac{1}{2}v_i\,v_i)v_{m,m} = \varrho\,b_i\,v_i + \sigma_{ij,j}\,v_i + \sigma_{ij}\,v_{i,j} - q_{i,i}$$

Tensor σ_{ij} is symmetric, so

$$\sigma_{ij}\,v_{i,j} = \frac{1}{2}(\sigma_{ij}\,v_{i,j} + \sigma_{ij}\,v_{i,j}) = \frac{1}{2}(\sigma_{ij}\,v_{i,j} + \sigma_{ji}\,v_{j,i}) = \sigma_{ij}\,\frac{1}{2}(v_{i,j} + v_{j,i}) = \sigma_{ij}\,D_{ij}$$

where D_{ij} is the rate of deformation tensor defined in (3.53). A little calculation yields:

$$\varrho\dot{u} + u(\dot{\varrho} + \varrho\, v_{m,m}) + v_i\,(\varrho\,\dot{v}_i - \varrho\, b_i - \sigma_{ij,j}) + \frac{1}{2}\, v_i\, v_i(\dot{\varrho} + \varrho\, v_{m,m}) = \sigma_{ij}\, D_{ij} - q_{i,i}$$

The terms in brackets vanish by the equation of motion (5.16) and equation of continuity (5.9), this equation is reduced to

$$\varrho\dot{u} = \sigma_{ij}\, D_{ij} - q_{i,i} \tag{5.24}$$

or in absolute form:

$$\varrho\dot{u} = \boldsymbol{\sigma} \cdot \mathbf{D} - \operatorname{div} \mathbf{q} \tag{5.25}$$

This is the energy equation in the deformed configuration.

5.6 Second Law of Thermodynamics: Clausius-Duhem Inequality

The first law of thermodynamics is a statement of the energy balance, which applies regardless of the direction in which the energy conversion between work and heat is assumed to occur. In real thermodynamic process we always have dissipation. The quantity of energy is conserved but an amount of it transforms into heat to the surrounding and can not be recovered. The second law of thermodynamics imposes restrictions on possible directions of thermodynamic processes. Work can be changed to heat, but the reverse process is impossible because of the inherent loss of usable heat when work is done, e.g. heat produced by friction of the system.

A state function, called the entropy of the system, is introduced as a measure of useful energy. We postulate that there exist two functions:

- Absolute temperature T ($T > 0$)
- Entropy S, or specific entropy s

  It means that the entropy is additive:

$$S = \int_V \varrho s dV \tag{5.26}$$

The entropy can change by interaction of the system with its surroundings through the heat transfer $dS^{(e)}$, and by irreversible changes $dS^{(i)}$ that take place inside the system:

$$dS = dS^{(e)} + dS^{(i)} \tag{5.27}$$

where:

$$dS^{(e)} = \frac{dQ}{T} \tag{5.28}$$

and

$$dS^{(i)} \geq 0 \qquad dS^{(i)} = 0 \ \text{ only for reversible processes} \tag{5.29}$$

The inequality:

$$dS^{(i)} = dS - dS^{(e)} > 0 \tag{5.30}$$

is a statement of the second law of thermodynamics for irreversible processes. Hence for any process:

$$\frac{dS}{dt} - \frac{1}{T}\frac{dQ}{dt} \geq 0 \tag{5.31}$$

Inequality (5.31) is known as the Clausius - Duhem inequality.

A simple example: why the heat flows always in the direction from the hot to the cold part of the body? Consider a body like a system of two parts with temperatures T_1 and T_2 respectively. In the heat flow process, some quantity dQ of heat flows from the first to the second part, then the entropy of the first part loses an amount $dS_1 = -dQ/T_1$, while the entropy of the second part gains an amount $dS_2 = dQ/T_2$. The entropy of body in this process must be

$$dS = dQ\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = dQ\,\frac{T_1 - T_2}{T_1 T_2} > 0$$

following from the second law of thermodynamics, from this we must have $T_1 > T_2$ because $T_1 > 0$ and $T_2 > 0$.

From (5.31) we get the global form of the second law of thermodynamics:

$$\frac{d}{dt}\int_V \varrho s\, dV + \int_S \frac{\mathbf{q}\cdot\mathbf{n}}{T}\, dS \geq 0 \tag{5.32}$$

Evaluating the material derivative in the first term, we have

$$\int_V (\varrho\,\dot{s} + \dot{\varrho}\,s + \varrho\,s\operatorname{div}\mathbf{v})\,dV + \int_S \frac{\mathbf{q}\cdot\mathbf{n}}{T}\,dS \geq 0$$

$$\int_V \varrho\,\dot{s}\,dV + \int_V s\,(\dot{\varrho} + \varrho\operatorname{div}\mathbf{v})\,dV + \int_S \frac{\mathbf{q}\cdot\mathbf{n}}{T}\,dS \geq 0$$

The second term vanishes according to the equation of continuity, then:

$$\int_V \varrho\dot{s}\,dV + \int_S \frac{\mathbf{q}\cdot\mathbf{n}}{T}\,dS \geq 0$$

By using the Gauss theorem, we obtain:

$$\int_V \varrho\dot{s}\,dV + \int_V \operatorname{div}\left(\frac{\mathbf{q}}{T}\right)\,dV \geq 0 \tag{5.33}$$

because V is arbitrary, then we have the local form of the second law of thermodynamic:

$$\varrho\dot{s} + \mathrm{div}\left(\frac{\mathbf{q}}{T}\right) = \varrho\dot{s} + \left(\frac{q_i}{T}\right)_{,i} \geq 0$$

$$\varrho\dot{s} + \frac{1}{T}\,q_{i,i} - \frac{1}{T^2}\,q_i\,T_{,i} \geq 0 \tag{5.34}$$

In Cartesian coordinate system, equation (5.34) reads:

$$\varrho\,\frac{ds}{dt} + \frac{1}{T}\left(\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3}\right) - \frac{1}{T^2}\left(q_1\,\frac{\partial T}{\partial x_1} + q_2\,\frac{\partial T}{\partial x_2} + q_3\,\frac{\partial T}{\partial x_3}\right) \geq 0$$

We can write (5.34) in the absolute form:

$$\varrho\dot{s} + \frac{1}{T}\,\mathrm{div}\mathbf{q} - \frac{1}{T^2}\;\mathbf{q}\cdot\mathrm{grad}T \geq 0 \tag{5.35}$$

If deformation is reversible (e.g., thermoelastic deformation), the entropy production rate $dS^{(i)}$ is equal to zero (see (5.30))

$$dS = dS^{(e)} = \frac{dQ}{T} \tag{5.36}$$

which means that the rate of entropy change is due to heat transfer only, and from this we have:

$$\varrho\,T\dot{s} = -\mathrm{div}\mathbf{q} \tag{5.37}$$

Combining the first law of thermodynamic (5.25) and (5.37) for reversible process, we have:

$$\dot{u} = \frac{1}{\varrho}\,\sigma_{ij}\,D_{ij} + T\dot{s} \tag{5.38}$$

We adopt the Fourier law for heat flow:

$$\mathbf{q} = -k\,\mathrm{grad}\,T \tag{5.39}$$

where k is the coefficient of heat conduction (with dimension $[k] = [J/(m\,K\,s)]$, K is symbol of degree kelvin). Here $k > 0$ because heat flows always in the direction from the hot to the cold region.

We have another relation relating the quantities appearring in this chapter. Multiplying the temperature change by the mass and specific heat capacities of the substances gives a value for the energy given off or absorbed during the process. Let c_v be the specific heat at constant strain (with dimension $[c] = [J/(kg\,K)]$), the calorimetric equation reads:

$$-\,\mathrm{div}\mathbf{q} = \varrho\,c_v\,\dot{T} \tag{5.40}$$

All equations obtained from Chapters 3, 4 and 5 are grouped in the following table:

Mechanics of continuous media -		Unknowns -	Number of unknowns	Number of equations
Kinematics	$\dot{u}_i = v_i$	$u_i,\, v_i$	6	3
Kinematics	$\varepsilon_{ij} = 1/2(u_{i,j} + u_{j,i})$	ε_{ij}	6	6
Kinematics	$D_{ij} = 1/2(v_{i,j} + v_{j,i})$	D_{ij}	6	6
Continuity	$\dot{\varrho} + \varrho\, v_{i,\,i} = 0$	ϱ	1	1
Kinetics	$\sigma_{ij,\,j} + \varrho\, b_i = \varrho\, \dot{v}_i$	σ_{ij}	6	3
Thermodynamics	$\varrho\dot{u} = \sigma_{ij}\, D_{ij} - q_{i,\,i}$	$u,\, q_i$	4	1
Thermodynamics	$\varrho\dot{s} + \frac{1}{T}\, q_{i\,,\,i} - \frac{1}{T^2}\, q_i\, T_{,\,i} \geq 0$	$s,\, T$	2	1
Fourier law	$q_i = -k\, T_{,\,i}$			3
Calorimetry	$-q_{i,\,i} = \varrho\, c_v\, \dot{T}$			1
31 unknowns, 25 equations			**31**	**25**
Constitutive eqns	$\sigma_{ij} = f_{ij}(\varepsilon_{kl}, T)$			6
31 unknowns, 31 equations			**31**	**31**

We have studied kinetics of deformation, the state of stress and basic laws of continuum mechanics and obtained a system of 25 equations for 31 unknowns. All these relations are valid for every continuum because in derivations we didn't mention any material. The 25 equations obtained are not sufficient to describe the response of specific material in loadings, because we know from the experience that e.g. the response of rubber is different than that of steel in tension test (see e.g. section (3.10)). Moreover, under different conditions of loading, the responses of the same material are also different. We need to find 6 more equations to close the system of equations in this table. They relate the stress and strain pole and depend on specific material:

$$\sigma_{ij} = f_{ij}\,(\varepsilon_{kl}) \tag{5.41}$$

These equations are called *constitutive equations* or *physical equations*. They describe the response of specific material on external loadings. We will discuss them in next chapters.

Chapter 6

Linear Elasticity

6.1 Uniaxial Case

From a block of material we cut out a cylindrical test specimen in figure (6.1). We apply a load **P** on ends, the bar elongates and an elongation $\Delta l = l - l_0$ can be measured. We can have a plot of magnitude of **P** of the load again its elongation. Such diagram contains useful information for the bar under consideration, but depends on the cross section S and the length l, then can not be directly used to predict, for example, the deformation of another bar with the same material but having different dimensions. So

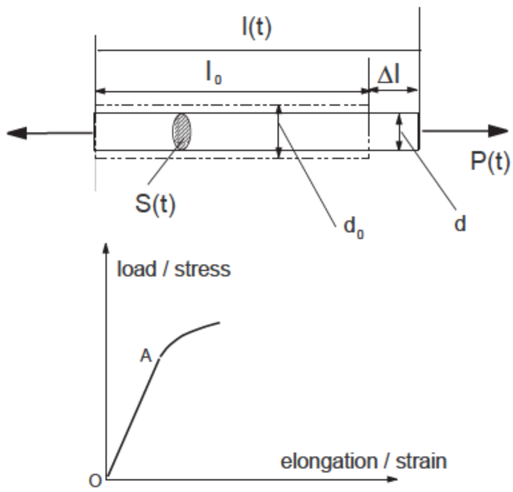

Figure 6.1: Uniaxial tension

we may plot the stress $\sigma = P/S_0$ (where S_0 is the undeformed cross section area) against the strain $\Delta l/l_0$ (l_0 is the initial length) and obtain a characteristic curve of the material. It does not then depend on the dimensions of the specimen. This curve is called a *stress-strain diagram*. When strain is small, the initial portion of the diagram OA is a line: the stress σ is directly proportional to the strain ε, and we may write:

$$\sigma = E\,\varepsilon \tag{6.1}$$

This relation is known as *Hooke's law*. The slope E of the line OA is called the *modulus of elasticity* or *Young's modulus*. Since the strain is a dimensionless quantity, the modulus E has the same units like the stress σ. Typical value of E for steel is around $200GPa$.

When stress is smaller than the value of stress at point A, the strain caused disappears when load is removed. The material is said to behave *elastically*. When a body deforms elastically under a load, it will revert to its original shape as soon as the load is removed. The stress at point A is called the *elastic limit*.

Note: We recall that the stress obtained by dividing the load P by the undeformed cross sectional area S_0 does not represent the true stress $\sigma = P/S(t)$. Also, instead of using the original value l_0, some scientists use the successive values of l to get the true strain. But because elastic strains are small, it does not matter whether the relations are expressed in terms of any strain measures (see Section 3.10) and stress measure (see Section 4.7).

In all engineering materials, the elongation produced by P is accompanied by a contraction in any transverse direction. If the bar is of circular cross section with an initial diameter d_0, then under some conditions, it will remain circular with diameter d. Let $\varepsilon^d = \Delta d/d$ be the lateral strain, then if the strain is small, the ratio:

$$\left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = const \tag{6.2}$$

or

$$\nu = -\frac{\varepsilon^d}{\varepsilon} \tag{6.3}$$

This value is called *Poisson's ratio*. For steel, the typical value of ν is 0.25.

We have considered only a specimen cut out from a block of material. If the value of Young's modulus E and Poisson's ratio ν depend on the orientation of the specimen, the material is called *anisotropic* with respect to elastic properties. Otherwise, when the specimens cut at different orientations at a small region of the block have the same properties, we say that in this small region, the material is called *isotropic*. When E and ν vary from point to point considered, the material is *imhomogeneous*, otherwise the material is said to be *homogeneous*.

Consider a homogeneous rod AB of uniform cross section (see Figure 6.2a), laying freely on a smooth horizontal surface. If the temperature of the rod is raised by ΔT, the rod elongates by an amount δ_T, which is proportional to both the temperature change and the length l, we have:

$$\delta_T = \alpha\,(\Delta T)\,l \tag{6.4}$$

where α is a constant characteristic of the material, called the *coefficient of thermal expansion*, measured in $[1/K]$, K is symbol of degree kelvil). A strain ε^T is associated with δ^T:

$$\varepsilon^T = \alpha\,\Delta T \tag{6.5}$$

and this is called a *thermal strain* caused by the change of temperature in the rod [3]. In this case, there is no stress associated with the thermal strain.

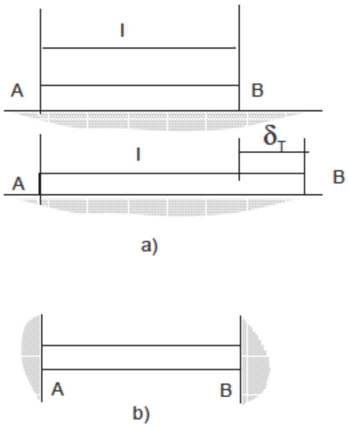

Figure 6.2: Problems with change of temperature

Let's now assume that the same rod AB is placed between two fixed supports like in figure 6.2b. If we raise the temperature by ΔT, the rod can not elongate because of the restraints, the elongation $\delta_T = 0$, then $\varepsilon^T = 0$. However, the supports will act on the rod and a state of stress σ^T is created in the rod.

Note: From Hooke's law, $\sigma = E\,\varepsilon^T = E\,\alpha\,\Delta T$. For steel, e.g. with $E = 280MPa$, $\Delta T = 100K$, the stress is

$$280 \times 10 \times 10^{-6} \times 100 = 280kPa$$

so this is comparable with mechanical loading in practice. The elongation is then $\Delta l = \alpha\,\Delta T\,l$.

We'll now introduce another important concept of *strain energy*. Consider a rod of length l and uniform cross section S. One end of the rod is attached to a fixed support, and the other end is subjected to *a slowly increasing* axial load $\mathbf{P}$. The work U done by the load $\mathbf{P}$ as the rod elongates by a small amount dx is:

$$dU = P\,dx \tag{6.6}$$

The expression obtained is equal to the element area of width dx on Figure 6.3. The work U done by the load as the rod undergoes a deformation x is thus:

$$U = \int_0^x P\,dx \tag{6.7}$$

This work results in the increase of some energy called the *strain energy* accumulated in the rod, so by definition strain energy $= U$. Dividing the strain energy by the volume $V = S\,l$ of the rod, using $\sigma_{xx} = P/S$, $dx/l = \varepsilon_{xx}$:

$$\text{strain energy density} = \int_0^{\varepsilon_{xx}} \sigma_{xx}\,d\varepsilon_{xx} \tag{6.8}$$

In elasticity $\sigma_{xx} = E\,\varepsilon_{xx}$:

$$\text{strain energy density} = \int_0^{\varepsilon_{xx}} \sigma_{xx}\,d\varepsilon_{xx} = \frac{E}{2}\,\varepsilon^2{}_{xx} = \frac{1}{2\,E}\,\sigma^2{}_{xx} = \frac{1}{2}\,\sigma_{xx}\,\varepsilon_{xx} \tag{6.9}$$

In the next section we study elastic behavior of homogeneous isotropic materials, the temperature dependence of elasticity, and thermal expansion.

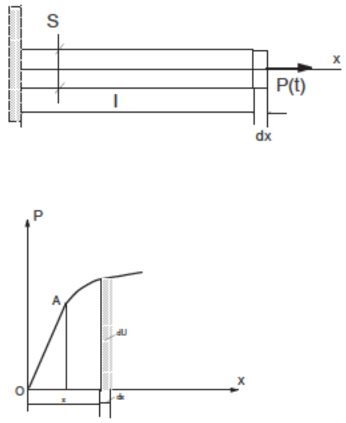

Figure 6.3:

6.2 General Derivation

A natural generalization of (6.1) is:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \tag{6.10}$$

We assume that the stress depends only on the strain and not on the history of that strain and this relation is *linear*. Also, the deformations are small, so we can use the infinitesimal tensor of strain ε_{ij}. When the C_{ijkl} are material constants, the material is said to be *linearly elastic*. Equations (6.10) imply that there exists an initial stress-free state and they are known as the *generalized Hooke's law*.

In general, the fourth-order tensor C_{ijkl} has $3^4 = 81$ components. It has however the following properties:

- $C_{ijkl} = C_{jikl}$ since the Cauchy stress tensor is symmetric $\sigma_{ij} = \sigma_{ji}$
- $C_{ijlk} = C_{jikl}$ since the strain tensor is symmetric $\varepsilon_{kl} = \varepsilon_{lk}$
- For an elastic material, it can be shown that we have also the reciprocal symmetry $C_{ijkl} = C_{klij}$ since $\dfrac{\partial^2 U}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} = \dfrac{\partial^2 U}{\partial\varepsilon_{kl}\partial\varepsilon_{ij}}$, U is the elastic strain energy defined later in (6.36)

As a result, the number of independent constants of C_{ijl} is reduced to 21. Equations (6.10) in matrix form in an Cartesian system of coordinate $O\,x_1\,x_2\,x_3$ read:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131}\\ C_{1122} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231}\\ C_{1133} & C_{2233} & C_{3333} & C_{3312} & C_{3323} & C_{3331}\\ C_{1112} & C_{2212} & C_{3312} & C_{1212} & C_{1223} & C_{1231}\\ C_{1123} & C_{2223} & C_{3323} & C_{1223} & C_{2323} & C_{2331}\\ C_{1131} & C_{2231} & C_{3331} & C_{1231} & C_{2331} & C_{3131}\end{pmatrix} \begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\end{pmatrix} \tag{6.11}$$

If in addition, we have a *plane of material symmetry*, e.g. $O\,x_2\,x_3$ then take the new system of coordinates as follows:

$$x_1^{'} = -x_1,\ x_2^{'} = x_2,\ x_3^{'} = x_3 \tag{6.12}$$

with the matrix (2.31) of the form:

$$Q_{ij} = \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{6.13}$$

In the new system of coordinate, only the first component of vector of displacement (3.6) changes its sign:

$$u_1^{'} = -u_1,\ u_2^{'} = u_2,\ u_3^{'} = u_3$$

Only these components of the strain tensor (3.56) change their signs: $\varepsilon_{21}^{'} = -\varepsilon_{21}$, $\varepsilon_{31}^{'} = -\varepsilon_{31}$. For a linearly elastic material, material symmetry with respect to that plane requires that the components of C_{ijkl} in the equation (6.10) [8]

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

be exactly the same as in the equation:

$$\sigma'_{ij} = C'_{ijkl}\,\varepsilon'_{kl}$$

under the change of coordinate (6.12), thus:

$$C'_{ijkl} = C_{ijkl} \tag{6.14}$$

Following (2.35), the components C'_{ijkl} of tensor in the new system of coordinate are:

$$C'_{ijkl} = Q_{im}\,Q_{jn}\,Q_{kp}\,Q_{lq}\,C_{mnpq} \tag{6.15}$$

then we obtain from (6.14) and (6.15):

$$C_{ijkl} = Q_{im}\,Q_{jn}\,Q_{kp}\,Q_{lq}\,C_{mnpq} \tag{6.16}$$

From this equation, we find that all C_{ijkl} with an odd number of the subscript 1 are zero:

$$C_{1112} = C_{1113} = C_{2212} = C_{2213} = C_{3312} = C_{3313} = 0 \tag{6.17}$$

The number of elastic constants are then reduced to 13.

We give another proof by considering the stress tensor. With such change of the coordinate system, the components σ_{33}, σ_{22} and σ_{23} of the stress tensor remain unchanged. Then we have for the component $\sigma'_{33} = \sigma_{33}$:

$$\sigma'_{33} = C_{1133}\varepsilon'_{11} + C_{2233}\varepsilon'_{22} + C_{3333}\varepsilon'_{33} + 2C_{3312}\varepsilon'_{12} + 2C_{3323}\varepsilon'_{23} + C_{3331}\varepsilon'_{31}$$

$$\sigma'_{33} = C_{1133}\varepsilon_{11} + C_{2233}\varepsilon_{22} + C_{3333}\varepsilon_{33} - 2C_{3312}\varepsilon_{12} + 2C_{3323}\varepsilon_{23} - 2C_{3331}\varepsilon_{31}$$

while in the "old" system:

$$\sigma_{33} = C_{1133}\varepsilon_{11} + C_{2233}\varepsilon_{22} + C_{3333}\varepsilon_{33} + 2C_{3312}\varepsilon_{12} + 2C_{3323}\varepsilon_{23} + C_{3331}\varepsilon_{31}$$

From $\sigma'_{33} = \sigma_{33}$, we have $C_{3312} = C_{3331} = 0$. Similarly we can show (6.17).

Hence, in this case the number of elastic constants is reduced to 13:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & C_{1123} & 0 \\ & C_{2222} & C_{2233} & 0 & C_{2223} & 0 \\ & & C_{3333} & 0 & C_{3323} & 0 \\ & & & C_{1212} & 0 & C_{1231} \\ & (symmetry) & & & C_{2323} & 0 \\ & & & & & C_{3131} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{pmatrix} \tag{6.18}$$

Further, if there is a second plane of elastic symmetry orthogonal to the first, then this second plane of symmetry also implies the symmetry about the third orthogonal plane. The material is then called *orthotropic*, and thus the number of independent constants for a linear elastic orthotropic material is 9:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & (symmetry) & & & C_{2323} & 0 \\ & & & & & C_{3131} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{pmatrix} \tag{6.19}$$

For an isotropic material, the elastic constants must be the same for all directions and (6.10) now becomes:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{pmatrix} \tag{6.20}$$

Thus, there are only two independent constants λ and μ, called *Lame's constants*. Since the strain is dimensionless, λ and μ have the same dimensions as the stress tensor. They can be determined from simple tests corresponding to simple states of stress. We can write tensor C_{ijkl} in the index form:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{6.21}$$

and the Hooke's law (6.6) for a linear isotropic elastic material becomes:

$$\sigma_{ij} = \lambda\, \varepsilon_{kk} \delta_{ij} + 2\mu\, \varepsilon_{ij} \tag{6.22}$$

Applying $i = j$ in (6.22) we obtain a relation for the first invariant of strain tensor (3.63) $I_\varepsilon = \varepsilon_{kk}$ in terms of the first invariant of Cauchy's stress tensor (4.26) $I_\sigma = \sigma_{kk}$:

$$\varepsilon_{kk} = \frac{\sigma_{kk}}{3\,\lambda + 2\,\mu} \tag{6.23}$$

Substituting this result into (6.22) and solving for ε_{ij} we have the inverse relation:

$$\varepsilon_{ij} = \frac{1}{2\mu} \left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\, \delta_{ij}\, \sigma_{kk} \right) \tag{6.24}$$

Equations (6.22) and (6.24) are the constitutive equations for a linearly elastic isotropic material. These equations have an important consequence: for an isotropic material, the principal directions of stress and strain tensors coincide. We can use this fact to derive the Hooke's law in an engineering way as follows.

If uniaxial tension is applied (see (6.1) and (6.2)) in the x_1-direction, the tensile strain is $\varepsilon_{11} = \sigma_{11}/E$ and causes lateral strains $\varepsilon_{22} = \varepsilon_{33} = -\nu/\varepsilon_{11}$, where ν is Poisson number. Consider the strain, ε_{11}, produced by a general stress state, $\sigma_{11}, \sigma_{22}, \sigma_{33}$. The stress, σ_{11}, causes a contribution $\varepsilon_{11} = \sigma_{11}/E$. The stresses σ_{22}, σ_{33} then cause lateral contractions $\varepsilon_{11} = \nu\sigma_{22}/E$ and $\varepsilon_{11} = \nu\sigma_{33}/E$. Since the stress-strain relation is linear, then superposition holds:

$$\varepsilon_{11} = \frac{1}{E} \left[\sigma_{11} - \nu \left(\sigma_{22} + \sigma_{33} \right) \right] \tag{6.25}$$

Similarly, for a pure shear test we see that shear strains are affected only by the corresponding shear stress, so

$$\sigma_{12} = 2\, G\, \varepsilon_{12} \tag{6.26}$$

where G is also a material constant called *the shear modulus*. From equations (6.20) we have $\sigma_{12} = 2\,\mu\,\varepsilon_{12}$, then:

$$G = \mu \tag{6.27}$$

Similar expressions apply for all directions:

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right] \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{33} + \sigma_{11}\right)\right] \\
\varepsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right] \\
\varepsilon_{12} &= \frac{\sigma_{12}}{2\,G} \\
\varepsilon_{23} &= \frac{\sigma_{23}}{2\,G} \\
\varepsilon_{31} &= \frac{\sigma_{31}}{2\,G}
\end{aligned} \tag{6.28}$$

For an isotropic material, the constants λ, $G = \mu$, E and ν are not all independent.

Considering a state of simple tension, only $\sigma_{11} = \sigma$, all other $\sigma_{ij} = 0$, then from (6.24) and (6.28),

$$E = \frac{\sigma_{11}}{\varepsilon_{11}} = \frac{\mu\,(3\,\lambda + 2\,\mu)}{\lambda + \mu} \tag{6.29}$$

and the Poisson's ratio:

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{\lambda}{2\,(\lambda + \mu)} \tag{6.30}$$

Another elastic constant, the *bulk modulus*, K, is defined by the relation between the volume strain and the mean stress. From (6.23) it follows that:

$$K = \frac{\sigma_{mm}}{\varepsilon_{kk}} = \lambda + \frac{2}{3}\,\mu \tag{6.31}$$

Any of the five elastic constants E, ν, λ, $\mu = G$, or K can be expressed in terms of two others as in the following table :

	μ, ν	ν, λ	μ, λ	K, λ	μ, E
λ	$\frac{2\,\mu\,\nu}{1 - 2\,\nu}$	-	-	-	$\frac{\mu\,(E - 2\,\nu)}{3\,\mu - E}$
μ	-	$\frac{\lambda\,(1 - 2\,\nu)}{2\,\nu}$	-	$\frac{3}{2}\,(K - \lambda)$	-
ν	-	-	$\frac{\lambda}{2\,(\lambda + \mu)}$	$\frac{\lambda}{3\,K - \lambda}$	$\frac{E - 2\,\mu}{2\,\mu}$
E	$2\,\mu\,(1 + \nu)$	$\frac{\lambda\,(1 + \nu)\,(1 - 2\,\nu)}{\nu}$	$\frac{\mu\,(3\,\lambda + 2\,\mu)}{\lambda + \mu}$	$\frac{9\,K\,(K - \lambda)}{3\,K - \lambda}$	-
K	$\frac{2\,\mu\,(1 + \nu)}{3\,(1 - 2\,\nu)}$	$\frac{\lambda\,(1 + \nu)}{3\,\nu}$	$\lambda + \frac{2}{3}\,\mu$	-	$\frac{\mu\,E}{3\,(3\,\mu - E)}$

	μ, K	ν, E	ν,K	K, E
λ	$K - \frac{2}{3}\mu$	$\frac{E\,\nu}{(1+\nu)\,(1-2\,\nu)}$	$\frac{3\,K\,\nu}{1+\nu}$	$\frac{3\,K\,(3\,K-E)}{9\,K-E}$
μ	-	$\frac{E}{2\,(1+\nu)}$	$\frac{3\,K\,(1-2\,\nu)}{2\,(1+\nu)}$	$\frac{3\,K\,E}{9\,K-E}$
ν	$\frac{3\,K-2\,\mu}{2\,(3\,K+\mu)}$	-	-	$\frac{3\,K-E}{6\,K}$
E	$\frac{9\,K\,\mu}{3\,K+\mu}$	-	$3\,K\,(1-2\,\nu)$	-
K	-	$\frac{E}{3\,(1-2\,\nu)}$	-	-

The equations (6.16) now can be represented in compact form as:

$$\varepsilon_{ij} = \frac{1}{E}\,[\,(1+\nu)\,\sigma_{ij} - \nu\,\sigma_{kk}\,\delta_{ij}\,] \tag{6.32}$$

and the reverse relation is:

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\,\varepsilon_{kk}\,\delta_{ij}\right] \tag{6.33}$$

Substituting the decomposition of strain (3.67) and stress (4.30) in spherical and deviatoric parts into equation (6.33) we find:

$$s_{ij} = \frac{E}{1+\nu}\,e_{ij} = 2\,G\,e_{ij} \tag{6.34}$$

and

$$\sigma_m = \frac{\sigma_{kk}}{3} = K\,\varepsilon_{kk} \tag{6.35}$$

where σ_m is called the *mean normal stress.* Hence, the distortion e_{ij} is produced by the deviator of stress s_{ij}, while the volume change ε_{kk} is produced by the mean normal stress σ_m. The equations (6.34) and (6.35) are independent each of other.

In the general case, the elastic strain per unit volume for small deformations is (see (6.9)):

$$U = \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij} = \frac{1}{2}\left(s_{ij} + \frac{1}{3}\,\sigma_{kk}\,\delta_{ij}\right)\left(e_{ij} + \frac{1}{3}\,\varepsilon_{mm}\,\delta_{ij}\right) \tag{6.36}$$

or

$$U = \frac{1}{2}\,s_{ij}e_{ij} + \frac{1}{6}\,\sigma_{kk}\varepsilon_{mm} \tag{6.37}$$

The first part is the elastic strain energy of shape changes:

$$U^{(1)} = \frac{1}{2}\,s_{ij}e_{ij} = \frac{1}{2}\,s_{ij}\,\frac{s_{ij}}{2G} = \frac{II_s}{2G} = \frac{1+\nu}{E}II_s \tag{6.38}$$

and second part is the elastic strain energy of volumes:

$$U^{(2)} = \frac{1}{6}\,\sigma_{kk}\,\varepsilon_{mm} = \frac{1}{6}\,I_\sigma\,I_\varepsilon = \frac{I_\sigma^2}{18K} = \frac{1-2\nu}{6E}\,I_\sigma^2 \tag{6.39}$$

Taking into account the the contribution of temperature, we assume that the strain tensor is a sum of two terms:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(\sigma)} + \varepsilon_{ij}^{(T)} \tag{6.40}$$

where:

- $\varepsilon_{ij}^{(\sigma)}$ - strain produced by stress pole.
- $\varepsilon_{ij}^{(T)}$ - strain produced by temperature.

For an isotropic material, the Hooke's law is:

$$\begin{aligned} \varepsilon_{ij}^{(\sigma)} &= \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}\,\sigma_{kk}\right) \\ \varepsilon_{ij}^{(T)} &= \alpha(T-T_0)\delta_{ij} \end{aligned} \tag{6.41}$$

where α is the coefficient of thermal expansion which appeared in (6.4). The constitutive equations in thermo-elastics read:

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}\,\sigma_{kk}\right) + \alpha(T-T_0)\delta_{ij} \\ \sigma_{ij} &= \lambda\,\delta_{ij}\,\varepsilon_{kk} + 2\mu\,\varepsilon_{ij} - (3\lambda+2\mu)\,\alpha\,\delta_{ij}\,(T-T_0) \end{aligned} \tag{6.42}$$

Let us consider a simple case of temperature influence on a rod

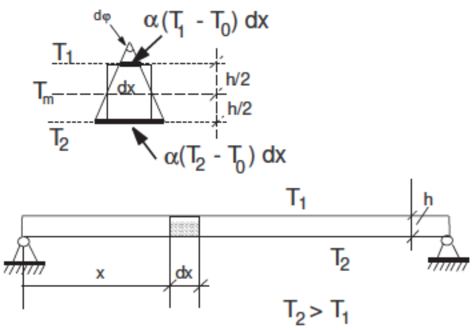

Figure 6.4: Temperature influence on construction

Problem 17 *For a road on figure (6.4), the temperature of the upper surface of the rod is T_1, and on the bottom is T_2, with $T_2 > T_1$. Assume that the temperature distribution is linear. Find the differential equation for the deflection of the rod.*

Solution:

From the assumption of linear distribution of temperature: $T_m = \dfrac{T_1+T_2}{2}$, drawing the free-body diagram of the portion of the rod, we find that the elongation: $\Delta l = \alpha\,(T_m - T_0)\,l = \alpha\left(\dfrac{T_1+T_2}{2}\right)l$, then:

$$h\,d\varphi = \alpha\,(T_2-T_0)\,dx - \alpha\,(T_1-T_0)\,dx$$

$$\frac{d\varphi}{dx} = \alpha\,\frac{(T_2-T_1)}{h}$$

The differential equation for elastic curve of the rod is:

$$\frac{d^2w}{dx^2} = \frac{M}{EJ} + \alpha\,\frac{(T_2-T_1)}{h} \tag{6.43}$$

where w is the deflection, M is the bending moment, E the modulus of eleasticity, and J the moment of inertia of the cross section about its neutral axis. The last term is the influence of temperature. Assume, for example, that $T_2 - T_1 = T_0\, x$, $\dfrac{d^2w}{dx^2} = \dfrac{\alpha\, T_0}{h}\, x$, then by integration we obtain $w = \dfrac{T_0 \alpha}{6h}\, x^3 + C_1 x + C_2$, the constants C_1 and C_2 can be found from the boundary conditions $w = 0$ for $x = 0$ and for $x = l$.

6.3 Equations of the Infinitesimal Theory of Elasticity

From Chapter 5, we know that for elastic process the rate of entropy change is due to heat transfer only (5.37):

$$\varrho\, T \dot{s} = -\mathrm{div}\mathbf{q} = -q_{i,\,i} \tag{6.44}$$

or

$$\varrho\, T \left(\frac{\partial s}{\partial \varepsilon_{ij}}\, \dot{\varepsilon_{ij}} + \frac{\partial s}{\partial T}\, \dot{T} \right) = -q_{i,\,i} \tag{6.45}$$

In thermodynamics these relations can be shown:

$$c_v = T\, \frac{\partial s}{\partial T}$$

$$\frac{\partial s}{\partial \varepsilon_{ij}} = -\frac{1}{\varrho}\, \frac{\partial \sigma_{ij}}{\partial T}$$

where c_v is the specific heat at constant strain introduced in (5.40). Then, equation (6.45) gives:

$$-\,T\, \frac{\partial \sigma_{ij}}{\partial T}\, \dot{\varepsilon}_{ij} + \varrho\, c_c\, \dot{T} = -a_{i,i} \tag{6.46}$$

From (6.42):

$$\frac{\partial \sigma_{ij}}{\partial T} = -(3\lambda + 2\mu)\, \alpha\, \delta_{ij}$$

and from the Fourier law for heat flow (5.39):

$$-q_{i,\,i} = -k\, T_{,\,i} = -k\, T_{,ii} T$$

hence equation (6.46) can be rewritten as:

$$k\, T_{,ii} = \varrho\, c_v\, \dot{T} + (3\lambda + 2\mu)\, \alpha\, T\, \dot{\varepsilon}_{kk} \tag{6.47}$$

where $T_{,ii} = \dfrac{\partial^2 T}{\partial x_1^2} + \dfrac{\partial^2 T}{\partial x_2^2} + \dfrac{\partial^2 T}{\partial x_3^2}$ is the Laplace operator. The equation (6.47) shows the coupling thermomechanical effects

Finally, we obtain the following system of equations:

Linear Thermoelasticity		Number of eqns
Kinematics	$\varepsilon_{ij} = 1/2(u_{i,\,j} + u_{j,\,i})$	6
Continuity	$\dot{\varrho} + \varrho\, \dot{u}_{i,\,i} = 0$	1
Equation of motion	$\sigma_{ij,\,j} + \varrho\, b_i = \varrho\, \ddot{u}_i$	3
Coupling effects	$k\, \Delta T = \varrho\, c_v\, \dot{T} + (3\lambda + 2\mu)\, \alpha\, T\, \dot{\varepsilon}_{kk}$	1
Hooke's Law	$\sigma_{ij} = \lambda\, \delta_{ij}\, \varepsilon_{kk} + 2\mu\, \varepsilon_{ij} - (3\lambda + 2\mu)\, \alpha\, \delta_{ij}\, (T - T_0)$	6

This is a system of 17 equations for 17 unknowns: ϱ, u_i, ε_{ij}, σ_{ij} and T. We also have to satisfy the initial and boundary conditions.

If on the boundary of a body some distributive forces $\mathbf{\Sigma}$ are applied, the we call them the *surface traction*. The vector of stress inside a body is defined by the Cauchy

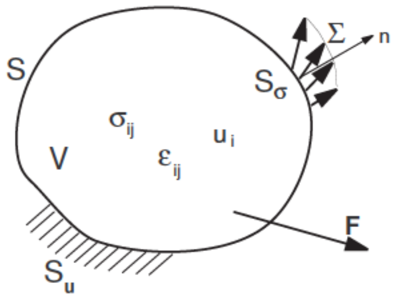

Figure 6.5: Boundary conditions

formula (4.22), then by continuity, at the boundary, we have:

$$\mathbf{\Sigma} = \boldsymbol{\sigma}\,\mathbf{n} \tag{6.48}$$

Equation (6.48) is called the stress boundary condition. In Cartesian coordinate system, this equation is:

$$\begin{aligned} \Sigma_1 &= \sigma_{11}\,n_1 + \sigma_{12}\,n_2 + \sigma_{13}\,n_3 \\ \Sigma_2 &= \sigma_{21}\,n_1 + \sigma_{22}\,n_2 + \sigma_{23}\,n_3 \\ \Sigma_3 &= \sigma_{31}\,n_1 + \sigma_{32}\,n_2 + \sigma_{33}\,n_3 \end{aligned} \tag{6.49}$$

In many problems of elasticity, the boundary conditions are such that only on one part of the boundary S_σ the surface traction is specified and on another part S_u displacements are specified (see Figure (6.5)).

6.4 Equations of the Infinitesimal Theory of Isothermal Elasticity

Consider the isothermal case, where the temperature $T = const$ and when motions of particles are small: every particle is in a small neighborhood of the initial state, the spacial and initial coordinate coincide $x_i = X_i$ (see (3.6)) and the magnitude of components $\partial u_i/\partial x_j$ is also small. Then the velocity (see Section 3.5):

$$v_i = \frac{du_i}{dt} = \left(\frac{\partial u_i}{\partial t}\right)_{x_i-fixed} + v_1\,\frac{\partial u_i}{\partial x_1} + v_2\,\frac{\partial u_i}{\partial x_2} + v_3\,\frac{\partial u_i}{\partial x_3} \approx \left(\frac{\partial u_i}{\partial t}\right)_{x_i-fixed} \tag{6.50}$$

because v_i is also small. Similarly for the acceleration:

$$a_i = \frac{d^2u_i}{dt^2} = \frac{dv_i}{dt} \approx \left(\frac{\partial^2 u_i}{\partial t^2}\right)_{x_i-fixed} \tag{6.51}$$

Everything discussed below is subjected to this assumption of linearization.

From (3.64) the relation between the differential volumes in the initial and actual configurations is

$$dV = (1 - \varepsilon_{kk})\, dV_0 \tag{6.52}$$

then the conservation of mass (equation of continuity) now is $dm = \varrho_0\, dV_0 = \varrho\, dV$ (ϱ_0 and ϱ are the density in the initial and actual states respectively) reads:

$$\varrho = \frac{\varrho_0}{(1 + \varepsilon_{kk})} \tag{6.53}$$

or using the Taylor's expansion we have approximately:

$$\varrho \approx (1 - \varepsilon_{kk})\, \varrho_0 \tag{6.54}$$

Neglecting the quantities of small orders, equation of motion (5.16) now becomes:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \varrho_0\, b_i = \varrho_0\, \frac{\partial^2 u_i}{\partial t^2} \tag{6.55}$$

where $\varrho_0\, b_i$ is the component of body force per unit volume, or:

$$\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \varrho_0\, b_1 &= \varrho_0\, \frac{\partial^2 u_1}{\partial t^2} \\
\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \varrho_0\, b_2 &= \varrho_0\, \frac{\partial^2 u_2}{\partial t^2} \\
\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \varrho_0\, b_1 &= \varrho_0\, \frac{\partial^2 u_3}{\partial t^2}
\end{aligned} \tag{6.56}$$

These three equations with 6 kinematic relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{6.57}$$

or

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1}; & \varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2}; & \varepsilon_{23} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\
\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3}; & \varepsilon_{31} &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)
\end{aligned} \tag{6.58}$$

and six equations obtained from the Hooke's law, e.g. of the form (6.22)

$$\begin{aligned}
\sigma_{11} &= \lambda\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\,\varepsilon_{11} \\
\sigma_{22} &= \lambda\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\,\varepsilon_{22} \\
\sigma_{33} &= \lambda\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\,\varepsilon_{33} \\
\sigma_{12} &= 2\mu\,\varepsilon_{12} \\
\sigma_{23} &= 2\mu\,\varepsilon_{23} \\
\sigma_{31} &= 2\mu\,\varepsilon_{31}
\end{aligned} \tag{6.59}$$

make a system of 15 equations to find 15 unknowns: three components of displacements u_1, u_2, u_3, six components of strain tensor ε_{11}, ε_{22}, ε_{33}, ε_{12}, ε_{23}, ε_{31} and six components of the stress tensor σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{23}, σ_{31}. On the boundary, these functions must satisfy the boundary conditions (e.g. (6.49) for stress) and compatibility equations for strain (3.68):

$$\begin{aligned}
\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} &= 2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1\,\partial x_2} \\
\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} &= 2\,\frac{\partial^2 \varepsilon_{23}}{\partial x_2\,\partial x_3} \\
\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} &= 2\,\frac{\partial^2 \varepsilon_{31}}{\partial x_3\,\partial x_1} \\
\frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} \\
\frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} \\
\frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2}
\end{aligned} \tag{6.60}$$

6.5 Navier's equations

We can combine Eqs. (6.22), (6.55), and (6.57) to obtain the equations of motion in terms of only the displacement components. Substituting the kinematic relation (6.57) into the Hooke's law (6.22):

$$\sigma_{ij} = \lambda\,\frac{1}{2}\,(u_{k,k} + u_{k,k})\,\delta_{ij} + 2\mu\,\frac{1}{2}(u_{i,j} + u_{j,i}) \tag{6.61}$$

$$\sigma_{ij} = \lambda\,u_{k,k}\,\delta_{ij} + \mu\,u_{i,j} + \mu\,u_{j,i}$$

Introducing these relations into (6.55) we obtain:

$$\frac{\partial(\lambda\,u_{k,k}\,\delta_{ij} + \mu\,u_{i,j} + \mu\,u_{j,i})}{\partial x_i} + \varrho_0\,b_i = \varrho_0\,\frac{\partial^2 u_i}{\partial t^2} \tag{6.62}$$

Denoting by

$$\theta = \varepsilon_{kk} = u_{k,k} = u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \tag{6.63}$$

we have

$$(\lambda+\mu)\frac{\partial \theta}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}\right) + \varrho_0\,b_i = \varrho_0\,\frac{\partial^2 u_i}{\partial t^2} \tag{6.64}$$

or in a compact form

$$(\lambda+\mu)\,\mathrm{grad\,div}\mathbf{u} + \mu\Delta\mathbf{u} + \varrho_0\,\mathbf{b} = \varrho_0\,\frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{6.65}$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is *Laplace operator* or *Laplacian.* These equations are known as the *Navier's equations.*

Equations (6.64) are a system of three equations to find three components u_1, u_2, u_3 of the displacement vector. Having u_i, we calculate the components of strain tensor by (6.58), then by the Hooke's law we have the components of the stress tensor.

Problem 18 *Find the Navier's equation in one dimensional case* $u_1 = u_1(x_1, t)$, $u_2 = 0$, $u_3 = 0$ *in the absence of body forces.*

Solution: We have from (6.64):

$$(\lambda + \mu)\frac{\partial \theta}{\partial x_1} + \mu \frac{\partial^2 u_1}{\partial x_1^2} = \varrho_0 \frac{\partial^2 u_1}{\partial t^2}$$

where $\theta = \partial u_1/\partial x_1$, thus we get the *simple wave equation*:

$$\frac{\partial^2 u_1}{\partial t^2} = c_1^2 \frac{\partial^2 u_1}{\partial x_1^2} \tag{6.66}$$

with

$$c_1 = \sqrt{\frac{\lambda + 2\,\mu}{\varrho_0}}$$

6.6 The Beltrami-Michelle compatibility equations

In the previous section, we have chosen the displacement components u_i as the basic unknowns. In static problems of elasticity, it is more convenient to first find the stress σ_{ij} from the equation of equilibrium (see (6.55)):

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \varrho_0\, b_i = 0 \tag{6.67}$$

Then from the Hooke's law we calculate the components of the strain. The solution obtained are not unique and we have to use the compatibility equation (6.60) to single out the correct solution.

Thus putting the Hooke's law (6.24) in compatibility equation (3.70), using (6.67) we obtain six equations for components of stress, called the *Beltrami-Michelle compatibility equations.*

We can also get these equations from the Navier's equations for static case. Differentiating (6.64) with respect to x_j:

$$\mu(\Delta u_i)_{,j} + (\lambda + \mu)\theta_{,ij} + (\varrho_0\, b_i)_{,j} = 0 \tag{6.68}$$

By changing index $i \to j; j \to i$, we get:

$$\mu(\Delta u_j)_{,i} + (\lambda + \mu)\theta_{,ji} + (\varrho_0\, b_j)_{,i} = 0 \tag{6.69}$$

Now add the equations (6.68) and (6.69):

$$2\mu\Delta\varepsilon_{ij} + 2(\lambda + \mu)\theta_{,ij} + [(\varrho b_i)_{,j} + (\varrho_0\, b_j)_{,i}] = 0$$

Denote by $\Theta = \sigma_{kk}$, then from this equation and Hooke's law we obtain:

$$\Delta\sigma_{ij} + \frac{2(\lambda + \mu)}{3\lambda + 2\mu}\,\Theta_{,ij} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}\,\Delta\Theta + [(\varrho_0\, b_i)_{,j} + (\varrho_0\, b_j)_{,i}] = 0 \tag{6.70}$$

This is the Beltrami-Michelle compatibility equations.

6.7 Some simple problems

1. *Uniaxial extension*

Consider a cylindrical bar of arbitrary cross-section, in which body forces are assumed to be absent. The bar is under the action of equal and opposite normal traction's distributed uniformly at its two end faces and the lateral surface is free from any surface traction (see figure 6.6):

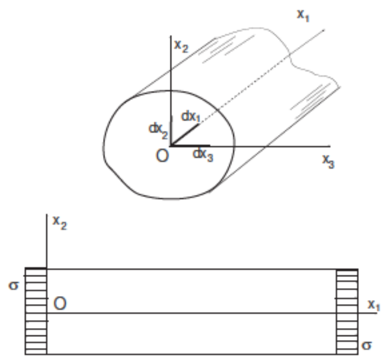

Figure 6.6: Uniaxial extension

Intuitively, assume that the stress state in that bar is [8]:

$$\sigma_{11} = \sigma,\ \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0 \tag{6.71}$$

This state of stress satisfies of course the equation of equilibrium (6.67) with the absence of volume forces $b_i = 0$ because all stresses are either constant or zero. We check now the bounding conditions for stress (6.49). For lateral surface, $\mathbf{n} = (0,\ n_2,\ n_3)$:

$$\begin{aligned} 0 = \Sigma_1 &= \sigma_{11}\, n_1 + \sigma_{12}\, n_2 + \sigma_{13}\, n_3 = \sigma\,.0 + 0.\, n_2 + 0.\, n_3 \\ 0 = \Sigma_2 &= \sigma_{21}\, n_1 + \sigma_{22}\, n_2 + \sigma_{23}\, n_3 = 0.\,0 + 0.\, n_2 + 0.\, n_3 \\ 0 = \Sigma_3 &= \sigma_{31}\, n_1 + \sigma_{32}\, n_2 + \sigma_{33}\, n_3 = 0.\, n_1 + 0.\, n_2 + 0.\, n_3 \end{aligned}$$

That is, the traction-free condition on the lateral surface is satisfied.

For the ends $x_1 = l$ or $x_1 = 0$, (l is the length of the bar): $\mathbf{n} = (\pm 1,\ 0,\ 0)$:

$$\begin{aligned} \pm\sigma = \Sigma_1 &= \sigma_{11}\, n_1 + \sigma_{12}\, n_2 + \sigma_{13}\, n_3 = \sigma\,.(\pm 1) + 0.\,0 + 0.\,0 \\ 0 = \Sigma_2 &= \sigma_{21}\, n_1 + \sigma_{22}\, n_2 + \sigma_{23}\, n_3 = 0.\,(\pm 1) + 0.\,0 + 0.\,0 \\ 0 = \Sigma_3 &= \sigma_{31}\, n_1 + \sigma_{32}\, n_2 + \sigma_{33}\, n_3 = 0.\,(\pm 1) + 0.\,0 + 0.\,0 \end{aligned}$$

so the traction conditions are also satisfied.

From the Hooke's law (6.32) we have the components of strain:

$$\begin{aligned} \varepsilon_{11} &= \frac{\sigma}{E} \\ \varepsilon_{22} &= \varepsilon_{33} = -\frac{\nu\,\sigma}{E} \\ \varepsilon_{12} &= \varepsilon_{23} = \varepsilon_{31} = 0 \end{aligned} \tag{6.72}$$

Since all strain components are constants, the equations of compatibility are automatically satisfied. From (6.58) we have:

$$\frac{\partial u_1}{\partial x_1} = \frac{\sigma}{E}$$

$$\frac{\partial u_2}{\partial x_2} = -\frac{\nu\,\sigma}{E}$$

$$\frac{\partial u_3}{\partial x_3} = -\frac{\nu\,\sigma}{E}$$

By integrating we obtain:

$$\begin{cases} u_1 = \dfrac{\sigma}{E}\,x_1 + f_1(x_2,\,x_3) \\ u_2 = -\dfrac{\nu\,\sigma}{E}\,x_2 + f_2(x_3,\,x_1) \\ u_3 = -\dfrac{\nu\,\sigma}{E}\,x_3 + f_3(x_1,\,x_2) \end{cases} \qquad (i)$$

where $f_1,\ f_2,\ f_3$ are integration functions. Substituting (i) in kinematic relations:

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0\,;\ \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = 0\,;\ \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) = 0$$

we get:

$$\begin{cases} \dfrac{\partial f_1(x_2,\,x_3)}{\partial x_2} = -\dfrac{\partial f_2(x_3,\,x_1)}{\partial x_1} = g_1(x_3) \\ \dfrac{\partial f_2(x_3,\,x_1)}{\partial x_3} = -\dfrac{\partial f_3(x_1,\,x_2)}{\partial x_2} = g_2(x_1) \\ \dfrac{\partial f_3(x_1,\,x_2)}{\partial x_1} = -\dfrac{\partial f_1(x_2,\,x_3)}{\partial x_3} = g_3(x_2) \end{cases} \qquad (ii)$$

where $g_1(x_3)$, $g_2(x_1)$ and $g_3(x_2)$ are integration functions. Integration of (ii) gives:

$$f_1(x_2,\,x_3) = g_1(x_3)\,x_2 + g_4(x_3) = -g_3(x_2)\,x_3 + g_5(x_2) \qquad (iii)$$
$$f_2(x_3,\,x_1) = g_2(x_1)\,x_3 + g_6(x_1) = -g_1(x_3)\,x_1 + g_7(x_3) \qquad (iv)$$
$$f_3(x_1,\,x_2) = g_3(x_2)\,x_1 + g_8(x_2) = -g_2(x_1)\,x_2 + g_9(x_1) \qquad (v)$$

From (iii):

$$g_1(x_3) = a_1\,x_3 + b_3;\ \ g_3(x_2) = -a_1\,x_2 + b_2,\ \ g_4(x_3) = -b_2\,x_3 + c_1,\ \ g_5(x_2) = b_1\,x_2 + c_1 \qquad (vi)$$

From (iv) and (vi):

$$g_2(x_1) = -a_1\,x_1 + b_1,\ \ g_6(x_1) = -b_3 x_1 + c_2,\ \ g_7(x_3) = b_1\,x_3 + c_2 \qquad (vii)$$

From (v), (vi) and (vii):

$$a_1 = 0,\ \ g_8(x_2) = -b_3\,x_2 + c_3,\ \ g_9(x_1) = b_2\,x_1 + c_3 \qquad (viii)$$

Then:

$$g_1(x_3) = b_3;\; g_2(x_1) = b_1;\; g_3(x_2) = b_2;\; g_4(x_3) = -b_2\, x_3 + c_1;\; g_5(x_2) = b_3\, x_2 + c_1$$

$$g_6(x_1) = -b_3\, x_1 + c_2;\; g_7(x_3) = b_1\, x_3 + c_2;\; g_8(x_2) = -b_1\, x_2 + c_3;\; g_9(x_1) = b_2\, x_1 + c_3$$

and:

$$\begin{aligned} f_1(x_2,\, x_3) &= b_3\, x_2 - b_2\, x_3 + c_1 && (ix)\\ f_2(x_3,\, x_1) &= b_1\, x_3 - b_3\, x_1 + c_2 && (x)\\ f_3(x_1,\, x_2) &= -b_1\, x_2 + b_2\, x_1 + c_3 && (xi) \end{aligned}$$

so that:

$$\begin{cases} u_1 = \dfrac{\sigma}{E}\, x_1 + b_3\, x_2 - b_2\, x_3 + c_1 \\[2ex] u_2 = -\dfrac{\nu\,\sigma}{E}\, x_2 + b_1\, x_3 - b_3\, x_1 + c_2 \\[2ex] u_3 = -\dfrac{\nu\,\sigma}{E}\, x_3 - b_1\, x_2 + b_2\, x_1 + c_3 \end{cases} \qquad (xii)$$

Assume that the origin O on the end cross section is fixed: for $x_1 = x_2 = x_3 = 0$, then $u_1 = u_2 = u_3 = 0$. From the above relation we have $c_1 = c_2 = c_3 = 0$. To prevent the rotation of the bar, two arbitrary of the three infinitesimal elements dx_1, dx_2, dx_3 should not rotate (see (3.60) and Figure 3.5). To prevent the rotation of the element dx_1 in the plane $x_1\, O\, x_2$ from rotating toward the axe x_2 we need that $\partial u_2/\partial x_1 = 0$; to prevent the rotation of the element dx_1 in the plane $x_1\, O\, x_3$ from rotating toward the axis x_3 we need $\partial u_3/\partial x_1 = 0$. Eliminating the possibility of rotation of the the element dx_2 in the plane $x_2\, O\, x_3$ toward the axis x_3 we need $\partial u_3/\partial x_2 = 0$. Substituting this relations into (xii) we find $b_1 = b_2 = b_3 = 0$. Hence, the displacement vector has two parts. The first part

$$\begin{cases} u_1 = \dfrac{\sigma}{E}\, x_1 \\[2ex] u_2 = -\dfrac{\nu\,\sigma}{E}\, x_2 \\[2ex] u_3 = -\dfrac{\nu\,\sigma}{E}\, x_3 \end{cases}$$

is corresponding to strain field (6.72) and the second part:

$$\begin{cases} u_1 = b_3\, x_2 - b_2\, x_3 + c_1 \\[2ex] u_2 = b_1\, x_3 - b_3\, x_1 + c_2 \\[2ex] u_3 = -b_1\, x_2 + b_2\, x_1 + c_3 \end{cases}$$

represents a rigid body displacement field.

2. *Uniaxial case with body force*

Consider now a bar with a length l standing vertically in gravitational field.

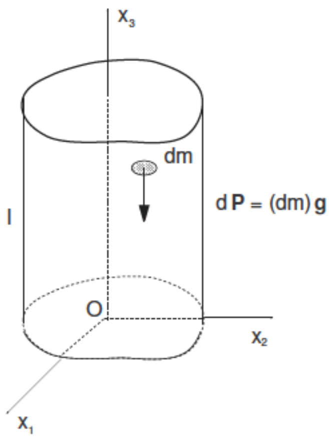

Figure 6.7:

From equation of equilibrium $\partial\sigma_{33}/\partial x_3 - \varrho\, g = 0$. On the upper end $x_3 = l$ we have $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. On lateral surface only $\sigma_{33} \neq 0$, all other components of stress equal zero. We find from equation of equilibrium:

$$\sigma_{33} = -\varrho g(l - x_3) \tag{6.73}$$

and all other $\sigma_{ij} = 0$.

From the Hooke's law we get the strain field:

$$\begin{cases} \varepsilon_{11} = \varepsilon_{22} = \dfrac{\nu}{E}\varrho g\,(l - x_3) \\ \varepsilon_{33} = -\dfrac{\varrho g(l - x_3)}{E} \\ \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0 \end{cases} \tag{6.74}$$

Integrating in the same way as in previous example, the part of the displacement corresponding to strain field is obtained as:

$$\begin{cases} u_1 = \dfrac{\nu}{E}\varrho g(l - x_3)\, x_1 \\ u_2 = \dfrac{\nu}{E}\varrho g(l - x_3)\, x_2 \\ u_3 = \dfrac{\varrho g}{2E}\left[(x_3)^2 - 2l\, x_3 + \nu(x_1^2 + x_2^2)\right] \end{cases} \tag{6.75}$$

Of course, any rigid body displacement field can be added to this without affecting the strain and stress field of the problem as discussed before.

2. *Torsion of a non-circular cylinder*

For cross-sections other than circular, the cross-sections will not remain planar. Assume that the component of the displacement u_3 does not depend on x_3 and $\sigma_{11} = \sigma_{12} = \sigma_{22} = \sigma_{33} = 0$. With the absence of the body forces, the equation of equilibrium (6.67) is:

$$\frac{\partial\sigma_{13}}{\partial x_3} = 0; \quad \frac{\partial\sigma_{23}}{\partial x_3} = 0; \quad \frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} = 0 \tag{6.76}$$

The first and the second equation shows that σ_{13} and σ_{23} do not depend on x_3, hence the distribution of shear stresses is similar for all cross-sections. Let φ is a function such that:

$$\sigma_{13} = \frac{\partial \varphi}{\partial x_2} \; ; \; \sigma_{23} = -\frac{\partial \varphi}{\partial x_1} \tag{6.77}$$

then the third equation of (6.76) satisfied automatically:

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} - \frac{\partial^2 \varphi}{\partial x_2 \partial x_1} \equiv 0,$$

The Beltrami-Michelle compatibility equations (6.70) reduce to:

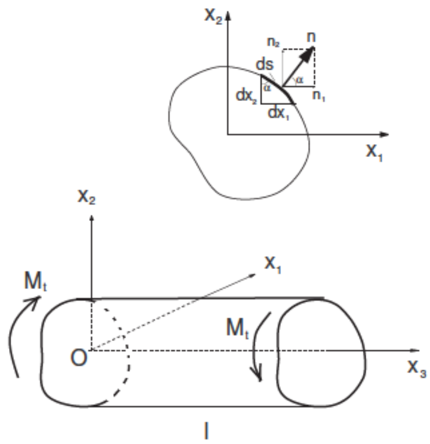

Figure 6.8:

$$\Delta\sigma_{13} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\sigma_{13} = 0; \qquad \Delta\sigma_{23} = 0 \tag{6.78}$$

Substituting (6.77) into (6.78) we get:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2}\right) = 0 \; ; \; \frac{\partial}{\partial x_2}\left(\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2}\right) \tag{6.79}$$

That means

$$\Delta\varphi = C = const \tag{6.80}$$

Now we check the boundary conditions. On lateral surface $\mathbf{n} = (n_1, n_2, 0)$ there is no surface traction, then from (6.49):

$$\sigma_{31}\, n_1 + \sigma_{32}\, n_2 = 0$$

This condition using (6.77) can be rewritten as:

$$\frac{\partial \varphi}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial \varphi}{\partial x_2}\frac{dx_2}{ds} = \frac{d\varphi}{ds} = 0$$

Then $\varphi = const$ on the boundary. Without loss of generality, we can choose this constant to be zero. Thus, to solve the problem of torsion we need to find a function φ, which satisfies (6.80) and is zero on the boundary.

The twisting moment is given by:

$$M_t = \int_S (x_1\,\sigma_{32} - x_2\,\sigma_{31})\,dxdy = -\int_S \left(x_1\,\frac{\partial\varphi}{\partial x_1} + x_2\,\frac{\partial\varphi}{\partial x_2} \right)\,dxdy = 2\int_S \varphi\,dS \qquad (6.81)$$

In the case of elliptical bar the bounding function is $(x_1/a)^2 + (x_2/b)^2 = 1$, where a is major radius and b-minor radius. Taking function φ of the form:

$$\varphi = m\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) \qquad (6.82)$$

then on lateral bounding we have $\varphi = 0$. Substituting (6.82) into (6.80) we calculate m, then:

$$\varphi = C\,\frac{a^2\,b^2}{2\,(a^2+b^2)}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) \qquad (6.83)$$

The constant C we can be found from (6.81):

$$C = -\frac{2\,M_t\,(a^2+b^2)}{\pi\,a^3\,b^3} \qquad (6.84)$$

and finally:

$$\varphi = -\frac{M_t}{\pi\,a\,b}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) \qquad (6.85)$$

Then stresses are obtained from (6.77):

$$\sigma_{13} = -\frac{2\,M_t}{\pi\,a\,b^3}\,x_2\ ;\ \sigma_{23} = \frac{2\,M_t}{\pi\,a^3\,b}\,x_1 \qquad (6.86)$$

The magnitude of shear stress on the cross-sectional plane is given by:

$$|\tau_s| = \sqrt{\sigma_{13}^2 + \sigma_{23}^2} = \frac{2\,M_t}{\pi\,a\,b}\sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}} \qquad (6.87)$$

and it takes the maximal value of shear stress at point $x_1 = 0$; $x_2 = b$:

$$|\tau_s|_{max} = \frac{2\,M_t}{\pi\,a\,b^2} \qquad (6.88)$$

Since $\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$, then from the Hooke's law $\dfrac{\partial u_1}{\partial x_1} = \dfrac{\partial u_2}{\partial x_2} = \dfrac{\partial u_3}{\partial x_3} = 0$, also we have $\sigma_{12} = 0$ then the kinematic relations (6.58) give:

$$\begin{aligned} 2\,\varepsilon_{12} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \\ 2\,\varepsilon_{13} &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \frac{\sigma_{13}}{\mu} \\ 2\,\varepsilon_{23} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = \frac{\sigma_{23}}{\mu} \end{aligned} \qquad (6.89)$$

Assume that $u_1 = -\theta\,x_2\,x_3$, $u_2 = \theta\,x_1\,x_3$ then the first equation in (6.89) is satisfied automatically. The second and the third equations have the form:

$$-\,\theta\,x_2 + \frac{\partial u_3}{\partial x_1} = \frac{\sigma_{13}}{\mu}\ ;\ \theta\,x_1 + \frac{\partial u_3}{\partial x_2} = \frac{\sigma_{23}}{\mu} \qquad (6.90)$$

or

$$\frac{\partial u_3}{\partial x_1} = \frac{\sigma_{13}}{\mu} + \theta\, x_2 \; ; \; \frac{\partial u_3}{\partial x_2} = \frac{\sigma_{23}}{\mu} - \theta\, x_1 \tag{6.91}$$

Differentiating the second equation in (6.89) with respect to x_2, the third equation there par rapport x_1, then subtracting, we obtain:

$$\frac{\partial^2 u_1}{\partial x_2 \partial x_3} - \frac{\partial^2 u_2}{\partial x_1 \partial x_3} = \frac{1}{\mu}\left(\frac{\partial \sigma_{13}}{\partial x_2} - \frac{\partial \sigma_{23}}{\partial x_1}\right) \tag{6.92}$$

When $u_1 = -\theta\, x - 2\, x_3$, $u_2 = \theta\, x_1\, x_3$, the left side of (6.92) is equal to $2\,\theta$ and the right side when taking into account (6.77) is equal to $(1/\mu)\,\Delta\varphi$. From (6.92) we have:

$$-\,2\,\mu\,\theta = C \tag{6.93}$$

In case of elliptical bar:

$$\theta = \frac{M_t\,(a^2 + b^2)}{\pi\, a^3\, b^3\, \mu} \tag{6.94}$$

Substituting (6.86), (6.94) into (6.91), after integrating we get:

$$u_3 = \frac{M_t\,(b^2 - a^2)}{\pi\, a^3\, b^3\, \mu}\, x_1\, x_2 \tag{6.95}$$

6.8 Plane stress and plane strain

6.8.1 Plane stress

The state of stress satisfying following conditions:

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0; \qquad b_3 = 0 \tag{6.96}$$

is called the *plane stress*. A very thin plate, its faces perpendicular to the x_3-axis, its lateral surface subjected to tractions that are independent of x_3, and its two end faces free from any surface traction, is approximately in a state of plane stress (see Figure 6.9).

From the Hooke's law (6.28), since $\sigma_{33} = 0$ then:

$$\varepsilon_{33} = -\frac{\nu}{E}\,(\sigma_{11} + \sigma_{22}) \tag{6.97}$$

$$\begin{cases} \varepsilon_{11} &= \dfrac{1}{E}\,(\sigma_{11} - \nu\,\sigma_{22}) \\ \varepsilon_{22} &= \dfrac{1}{E}\,(\sigma_{22} - \nu\,\sigma_{11}) \\ \varepsilon_{12} &= \dfrac{1+\nu}{E}\,\sigma_{12} \end{cases} \tag{6.98}$$

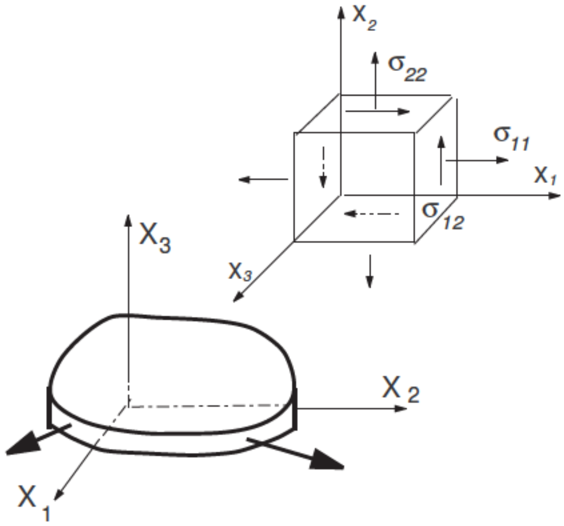

Figure 6.9: Plane Stress

6.8.2 Plane strain

When the displacement vector satisfies conditions:

$$u_1 = u_1(x_1, x_2);\ u_2 = u_2(x_1, x_2);\ u_3 = 0 \tag{6.99}$$

the strain tensor can be calculated as:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{6.100}$$

then

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1};\ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2};\ \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{6.101}$$

We see that the components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ do not depend on the coordinate x_3, while components $\varepsilon_{23} = \varepsilon_{31}, \varepsilon_{33} = 0$. Such state is called the *plane strain*. This state appears e.g. in a prismatic bar that has a uniform cross-section with its normal in the axial direction, which we take to be the x_3 axis (see Figure 6.10). The cross-sections are perpendicular to the lateral surface and parallel to the x_1x_2 plane. On its lateral surfaces, the surface tractions are also uniform with respect to the axial direction and have no axial (i.e., x_3) components. Its two end faces (here $x = 0$ and $x = l$) are prevented from axial displacements but are free to move in other directions (e.g., constrained by frictionless planes). The dilatation (3.64) has the form:

$$\theta_1 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \tag{6.102}$$

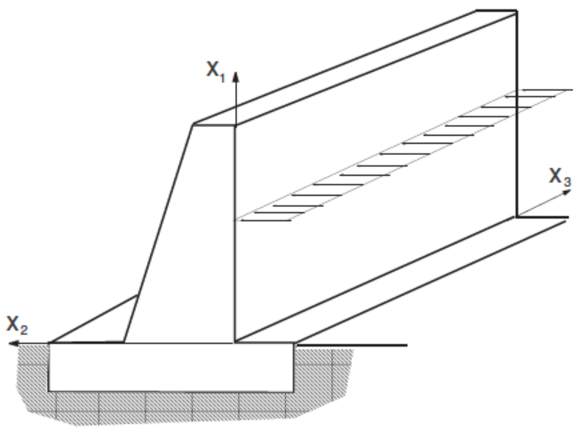

Figure 6.10: Plane Strain

The Hooke's law is:

$$\begin{cases} \varepsilon_{11} = \dfrac{1+\nu}{E}\sigma_{11} - \dfrac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \\ \varepsilon_{22} = \dfrac{1+\nu}{E}\sigma_{22} - \dfrac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \\ \varepsilon_{33} = \dfrac{1+\nu}{E}\sigma_{33} - \dfrac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \\ \varepsilon_{12} = \dfrac{1+\nu}{E}\sigma_{12} \\ \varepsilon_{13} = \dfrac{1+\nu}{E}\sigma_{13} \\ \varepsilon_{23} = \dfrac{1+\nu}{E}\sigma_{23} \end{cases} \tag{6.103}$$

From (6.83) we have:

$$\sigma_{33} = \nu(\sigma_{11}+\sigma_{22}); \ \sigma_{13} = \sigma_{23} = 0 \tag{6.104}$$

Substituting (6.104) into the third equation of (6.103):

$$\begin{cases} \varepsilon_{11} = \dfrac{1}{E_1}(\sigma_{11} - \nu_1\sigma_{22}) \\ \varepsilon_{22} = \dfrac{1}{E_1}(\sigma_{22} - \nu_1\sigma_{11}) \\ \varepsilon_{12} = \dfrac{1+\nu_1}{E_1}\sigma_{12} \end{cases} \tag{6.105}$$

where

$$E_1 = \frac{E}{1-\nu^2}, \ \nu_1 = \frac{\nu}{1-\nu} \tag{6.106}$$

Comparing (6.105) and (6.98), we can say that from the mathematical point of view, there is no difference between the plane stress and the plane strain.

6.8.3 Governing equations of Plane Elasticity

In this section we summarize the governing equations of plane elasticity, for both plane stress and plane strain. The stress-strain relations can be rewritten as [2]:

$$\varepsilon_{ij} = \frac{1}{2\,\mu}\left(\sigma_{ij} - \frac{3-\kappa}{4}\,\sigma_{kk}\,\delta_{ij}\right) \qquad \text{(i,j 1,2)} \tag{6.107}$$

where κ is the Kolosov constant defined by:

$$\kappa = \begin{cases} 3 - 4\,\nu & \text{for plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases} \tag{6.108}$$

There are two Cauchy equations of equilibrium:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \varrho b_1 &= 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \varrho b_2 &= 0 \end{cases} \tag{6.109}$$

and one compatibility equation:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \tag{6.110}$$

Using the compatibility equation, the Hooke's law and equations of equilibrium when there are no body forces, we can show that:

$$\Delta(\sigma_{11} + \sigma_{22}) = 0 \tag{6.111}$$

where $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplacian operator.

6.8.4 The Airy stress function

With the absence of body force, we have the following three equations for three components of stress:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} &= 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} &= 0 \\ \Delta(\sigma_{11} + \sigma_{22}) &= 0 \end{cases} \tag{6.112}$$

Let Φ be a function with such properties:

$$\begin{aligned} \sigma_{11} &= \frac{\partial^2 \Phi}{\partial x_2^2} \\ \sigma_{22} &= \frac{\partial^2 \Phi}{\partial x_1^2} \\ \sigma_{12} &= -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \end{aligned} \tag{6.113}$$

Then Φ satisfies the *biharmonic equation*:

$$\Delta\Delta\Phi \equiv \frac{\partial^4\Phi}{\partial x_1^4} + 2\,\frac{\partial^4\Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\Phi}{\partial x_2^4} = 0 \tag{6.114}$$

and is known as the *Airy stress function.* Any function that satisfies this biharmonic equation (6.114) generates a possible solution for a plane elastic-static problem.

6.8.5 Resume

We give here a brief resume of stress and strain fields in plane elasticity:

	Plane Stress	Plane Strain
Stress	$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ $\sigma_{11}, \sigma_{12}, \sigma_{22}$ may have non zero values	$\sigma_{13} = \sigma_{23} = 0$ $\sigma_{11}, \sigma_{12}, \sigma_{22}, \sigma_{33}$ may have non zero values
Strain	$\varepsilon_{13} = \varepsilon_{23} = 0$ $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22}, \varepsilon_{33}$ may have non zero values	$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$ $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22}$ may have non zero values

6.9 Solutions of plane problems in Cartesian coordinates

The solution of plane problems when there are no body forces is reduced to the integration of the biharmonic equation (6.114)

$$\Delta\Delta\Phi \equiv \frac{\partial^4\Phi}{\partial x_1^4} + 2\,\frac{\partial^4\Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\Phi}{\partial x_2^4} = 0 \tag{6.115}$$

with the boundary conditions (6.49):

$$\begin{aligned} \sigma_{11}\, n_1 + \sigma_{12}\, n_2 + \sigma_{13}\, n_3 &= \Sigma_1 \\ \sigma_{21}\, n_1 + \sigma_{22}\, n_2 + \sigma_{23}\, n_3 &= \Sigma_2 \end{aligned} \tag{6.116}$$

Solutions in form of polynomials are used widely to solve problems of beams [10]. First we consider a polynomial of the second order:

$$\Phi = \frac{A_2}{2}\,(x_1)^2 + B_2\, x_1\, x_2 + \frac{C_2}{2}\,(x_2)^2 \tag{6.117}$$

where A_2, B_2 and C_2 are constant. It is easy to check that (6.117) satisfies (6.115). Then from (6.113) we obtain:

$$\sigma_{11} = \frac{\partial^2\Phi}{\partial x_2^2} = C_2, \quad \sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2} = A_2, \quad \sigma_{12} = -\frac{\partial^2\Phi}{\partial x_1\, \partial x_2} = -B_2 \tag{6.118}$$

All three stresses are constant throughout the body and in the case of rectangular plate (Figure 6.11) we have a uniform tension (or compression depending on the sign of the constants) and a uniform shear.

Consider now a stress function in the form of a polynomial of the third order:

$$\Phi_3 = \frac{A_3}{3\cdot 2}\,(x_1)^3 + \frac{B_3}{2}\,(x_1)^2\, x_2 + \frac{C_3}{2}\, x_1\,(x_2)^2 + \frac{D_3}{3\cdot 2}\,(x_2)^3 \tag{6.119}$$

This also satisfies (6.103) for every constants A_3, B_3, C_3, D_3 and we have the stress field:

$$\sigma_{11} = C_3 x_1 + D_3 x_2\,;\ \sigma_{22} = A_3 x_1 + B_3 x_2\,;\ \sigma_{12} = -(B_3 x_1 + C_3 x_2) \tag{6.120}$$

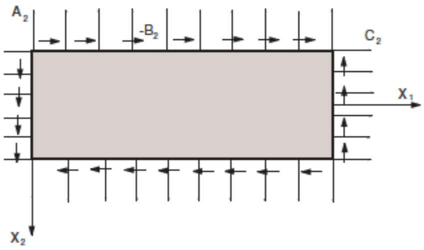

Figure 6.11:

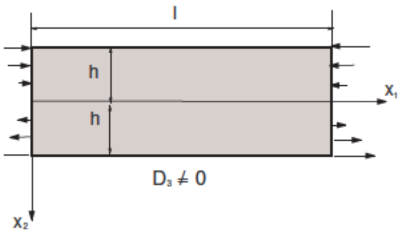

Figure 6.12:

When only $D_3 \neq 0$ as in Figure 6.12 we obtain pure bending.

In Figure 6.13 we present the case in which all coefficients except B_3 are equal to zero. Along the side $x_2 = -h$ we have a uniform compressive stresses, along $x_2 = h$: uniform tensile stresses. The side $x_1 = 0$ is free of traction and shearing stresses are distributed on $x_1 = l$.

For the stress function in the form of a fourth order polynomial:

$$\Phi_4 = \frac{A_4}{4 \cdot 3}(x_1)^4 + \frac{B_4}{3 \cdot 2}(x_1)^3 x_2 + \frac{C_4}{2}(x_1)^2(x_2)^2 + \frac{D_4}{3 \cdot 2}x_1(x_2)^3 + \frac{E_4}{4 \cdot 3}(x_2)^4 \qquad (6.121)$$

then by substituting (6.121)into equation (6.115), we find that (6.115) is satisfied only when:

$$E_4 = -(2\,C_4 + A_4) \qquad (6.122)$$

The stress field calculated from (6.113) is:

$$\begin{aligned} \sigma_{11} &= C_4\,(x_1)^2 + D_4\,x_1 x_2 - (2C_4 + A_4)(x_2)^2 \\ \sigma_{22} &= A_4\,(x_1)^2 + B_4\,x_1 x_2 + C_4\,(x_2)^2 \\ \sigma_{12} &= -(B_4/2)\,(x_1)^2 - 2C_4\,x_1 x_2 - (D_4/2)\,(x_2)^2 \end{aligned} \qquad (6.123)$$

Figure (6.14) presents the distribution of stresses on bounding of a rectangular plate, when only $D_4 \neq 0$, the stress field is follows:

$$\begin{aligned} \sigma_{11} &= D_4\,x_1 x_2 \\ \sigma_{22} &= 0 \\ \sigma_{12} &= -(D_4/2)\,(x_2)^2 \end{aligned} \qquad (6.124)$$

and we have uniformly distributions of shearing stresses on $x_2 = \pm h$. On the two ends $x_1 = 0$ and $x_1 = l$ shearing forces are distributed according to the parabolic law. Normal

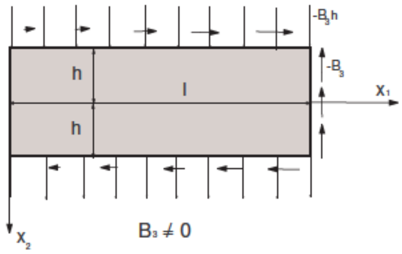

Figure 6.13:

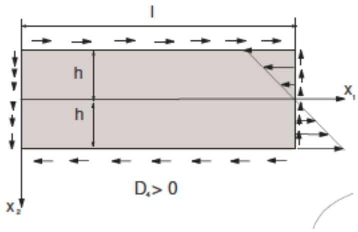

Figure 6.14:

traction on the end $x = l$ is linear, on the end $x = 0$ is equal to zero. Consider a stress function in a form of a polynomial of the fifth order:

$$\Phi_5 = \frac{A_5}{5\cdot 4}(x_1)^5 + \frac{B_5}{4\cdot 3}(x_1)^4 x_2 + \frac{C_5}{3\cdot 2}(x_1)^3(x_2)^2 + \frac{D_5}{3\cdot 2}(x_1)^2(x_2)^3 + \tag{6.125}$$
$$+ \frac{E_5}{4\cdot 3}x_1(x_2)^4 + \frac{F_5}{5\cdot 4}(x_2)^5$$

Equation (6.115) is satisfied only when the following relations are satisfied:

$$E_5 = -(2C_5 + 3A_5)\,;\; F_5 = -(1/3)(B_5 + 2D_5) \tag{6.126}$$

now the stress field is:

$$\begin{aligned}
\sigma_{11} &= (C_5/3)(x_1)^3 + D_5(x_1)^2 x_2 - (2C_5 + 3A_5)\,x_1(x_2)^2 - (1/3)(B_5 + 2D_5)(x_2)^3 \\
\sigma_{22} &= A_5(x_1)^3 + B_5(x_1)^2 x_2 + C_5\,x_1(x_2)^2 + (D_5/3)(x_2)^3 \\
\sigma_{12} &= -(1/3)B_5(x_1)^3 - C_5(x_1)^2 x_2 - D_5\,x_1(x_2)^2 + (1/3)(2C_5 + 3A_5)(x_2)^3
\end{aligned} \tag{6.127}$$

Hence, only coefficients A_5, B_5, C_5 and D_5 are arbitrary. In a special case when only

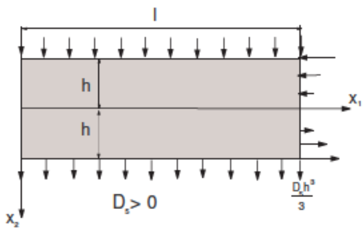

Figure 6.15:

$D_5 \neq 0$, then the stress field is:

$$
\begin{aligned}
\sigma_{11} &= D_5\,[(x_1)^2 x_2 - 2/3(x_2)^3] \\
\sigma_{22} &= (1/3)D_5(x_2)^3 \\
\sigma_{12} &= -D_5\,x_1(x_2)^2
\end{aligned}
\tag{6.128}
$$

and the distribution of normal forces is presented on Figure 6.15 while the distribution of shear forces is on Figure 6.16. Since the differential equation (6.115) is linear, then sum of

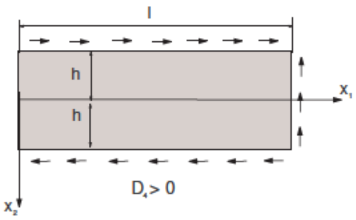

Figure 6.16:

its several solutions is also a solution. We can use the superposition method to find new solutions of (6.115).

6.9.1 Solution by Polynomials

Consider a narrow rectangular beam with unit width on Figure 6.17. The beam is supported at two ends and is bending by a uniform load of intensity q. The boundary conditions are:

- at upper edge $x_2 = -h$, $\mathbf{n} = (0, -1)$:

$$\sigma_{12} = 0, \;\; \sigma_{22} = -q \tag{6.129}$$

- at lower edge $x_2 = h$, $\mathbf{n} = (0, 1)$:

$$\sigma_{12} = 0, \;\; \sigma_{22} = 0 \tag{6.130}$$

- at $x = \pm l$, $\mathbf{n} = (\pm 1, 0)$. The strict boundary conditions (6.116) are:

$$\sigma_{11} = 0, \;\; \pm\sigma_{12} = \Sigma_2 \tag{6.131}$$

Using the stress functions in form of a polynomial, we get the exact solution only in the case when the surface forces are exactly as given before. We often use in solving problem of elasticity the so-called Saint Venant's principle. It says that the effect of the change in the boundary condition to a statically equivalent condition is local; that is, the solutions obtained with the two sets of boundary conditions are approximately the same at points sufficiently far from the points where the elasticity boundary conditions are replaced with statically equivalent boundary conditions. At a distance larger than the characteristic dimension of the portion mentioned, the stress distribution is the same whether the body is loaded as before. This statement can be applied to practically any type of load. In Figure 6.18, the value of normal stress at distance greater than b is nearly uniform.

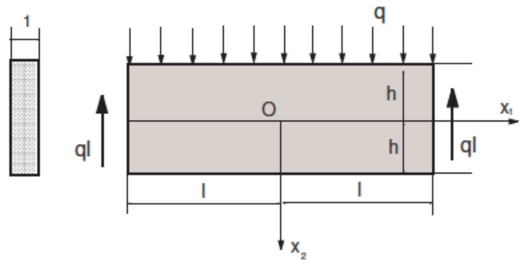

Figure 6.17:

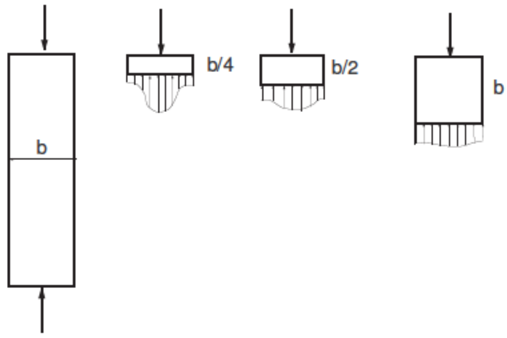

Figure 6.18: Saint Venant's principle

We should keep in mind two important points when applying this principle [3]: 1. The actual loading and the loading used to compute the stress must be *statically equivalent*, and 2. Stresses can not be computed in this manner in the immediate vicinity of the points of application of the loads. In our problem, use of the Saint Venant's principe gives the approximate conditions on the ends $x = \pm l$ of rectangular plate:

$$\int_{-h}^{h} \sigma_{12}\, dx_2 = \mp ql, \quad \int_{-h}^{h} \sigma_{11}\, dx_2 = 0, \quad \int_{-h}^{h} \sigma_{11}\, x_2\, dx_2 = 0 \qquad (6.132)$$

The last two conditions in (6.132) state that there are no longitudinal forces and bending moment at the end of the plate.

We will build now step by step a biharmonic stress function for our problem. We know already from the problem of bending of beam in the course of *Strength of Material*:

$$\begin{aligned} \sigma_{11} &= \frac{q(l^2 - x_1^2)\, x_2}{2J} \\ \sigma_{12} &= -\frac{qx_1}{2J}(h^2 - x_2^2) \end{aligned} \qquad (6.133)$$

where $J = 2h^3/3$ is the moment of inertia of the cross-section (with unit width) with respect to neutral axis. Then assume that:

$$\begin{aligned} \sigma_{11} &= Ax_2 + Bx_1^2 x_2 \\ \sigma_{12} &= Cx_1 + Dx_1 x_2^2 \end{aligned} \qquad (6.134)$$

Since $\sigma_{11} = \partial^2\Phi/\partial x_2^2$:

$$\frac{\partial^2\Phi}{\partial x_2^2} = Ax_2 + Bx_1^2x_2$$

After integrating we obtain:

$$\frac{\partial\Phi}{\partial x_2} = Ax_2^2/2 + Bx_1^2x_2^2/2 + f_1(x_1)$$

and now

$$\Phi = A\,\frac{x_2^3}{6} + B\,x_1^2\,\frac{x_2^3}{6} + f_1(x_1)\,x_2 + f_2(x_1) \tag{6.135}$$

Calculating the derivative and comparing it with the second equation of (6.133)

$$\frac{\partial^2\Phi}{\partial x_1\partial x_2} = Bx_1x_2^2 + f_1^{'}(x_1) \Rightarrow -Bx_1x_2^2 - f_1^{'}(x_1) = Cx_1 + Dx_1x_2^2$$

then

$$D = -B;\; f_1(x_1) = -Cx_1^2/2 + E$$

and

$$\Phi = A\,\frac{x_2^3}{6} + B\,x_1^2\,\frac{x_2^3}{6} - \frac{C}{2}\,x_1^2\,x_2 + Ex_2 + f_2(x_1) \tag{6.136}$$

The linear term Ex_2 does not influence on stress, and our function (6.136) now is still not biharmonic, so we add to it another function $\psi(x_1, x_2)$ and we demand that $\Delta\Delta\Phi = 0$, so that:

$$\Phi = A\,\frac{x_2^3}{6} + B\,x_1^2\,\frac{x_2^3}{6} - \frac{C}{2}\,x_1^2\,x_2 + Ex_2 + f(x_1) + \psi(x_1, x_2) \tag{6.137}$$

From (6.137) we have:

$$\begin{aligned}
\frac{\partial\Phi}{\partial x_1} &= Bx_1x_2^3/3 - Cx_1x_2 + f^{'}(x_1) + \frac{\partial\psi}{\partial x_1}\\
\frac{\partial^2\Phi}{\partial x_1^2} &= Bx_2^3/3 - Cx_2 + f''(x_1) + \frac{\partial^2\psi}{\partial x_1^2}\\
\frac{\partial^3\Phi}{\partial x_1^3} &= f'''(x_1) + \frac{\partial^3\psi}{\partial x_1^3}\\
\frac{\partial^4\Phi}{\partial x_1^4} &= f^{(IV)}(x_1) + \frac{\partial^4\psi}{\partial x_1^4}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2\Phi}{\partial x_1\partial x_2} &= Bx_1x_2^2 - Cx_1 + \frac{\partial^2\psi}{\partial x_1\partial x_2}\\
\frac{\partial^3\Phi}{\partial x_1^2\partial x_2} &= Bx_2^2 - C + \frac{\partial^3\psi}{\partial x_1^2\partial x_2}\\
\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} &= 2Bx_2 + \frac{\partial^4\psi}{\partial x_1^2\partial x_2^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x_2} &= Ax_2^2/2 + Bx_1^2x_2^2/2 - Cx_1^2/2 + \frac{\partial \psi}{\partial x_2} \\
\frac{\partial^2 \Phi}{\partial x_2^2} &= Ax_2 + Bx_1^2x_2 + \frac{\partial^2 \psi}{\partial x_2^2} \\
\frac{\partial^3 \Phi}{\partial x_2^3} &= A + Bx^2 + \frac{\partial^3 \psi}{\partial x_2^3} \\
\frac{\partial^4 \Phi}{\partial x_2^4} &= \frac{\partial^4 \psi}{\partial y^4}
\end{aligned}
$$

$$
\Delta\Delta\Phi \equiv \frac{\partial^4 \Phi}{\partial x_1^4} + 2\,\frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = f^{(IV)}(x_1) + \frac{\partial^4 \psi}{\partial x_1^4} + 4Bx_2 + 2\,\frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \psi}{\partial x_2^4} = 0
$$

Assume that $f^{(IV)}(x_1) = 0$, we get the following equation for ψ:

$$
\frac{\partial^4 \psi}{\partial x_1^4} + 2\,\frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \psi}{\partial x_2^4} = -4Bx_2
$$

The simplest solution of this equation is:

$$
\psi = \frac{F}{24}\,x_1^4x_2 + \frac{H}{120}\,x_2^5
$$

By substituting this relation to Φ and checking the relations between the constants for Φ to be a harmonic function we get

$$
H = -4B - F
$$

and finally the function:

$$
\Phi = \frac{A}{6}\,x_2^3 + \frac{B}{6}\,x_1^2\,x_2^3 - \frac{C}{2}\,x_1^2x_2 + Ex_2 + f(x_1) + \frac{H}{24}\,x_1^4x_2 - \frac{4B+F}{120}\,x_2^5 \tag{6.138}
$$

which satisfies $\Delta\Delta\Phi = 0$. The stress field can be calculated:

$$
\begin{cases}
\sigma_{11} = Ax_2 + Bx_1^2x_2 - \dfrac{4B+F}{6}\,x_2^3 \\[2ex]
\sigma_{12} = -Bx_1x_2^2 + Cx_1 - \dfrac{F}{6}\,x_1^3 \\[2ex]
\sigma_{22} = \dfrac{B}{3}\,x_2^3 - Cx_2 + f''(x_1) + \dfrac{F}{2}\,x_1^2x_2
\end{cases} \tag{6.139}
$$

To find the constants in (6.139) we use the bounding conditions:

1. For the edge $x - 2 = h$, $\sigma_{22} = 0$ for every x_1 then

$$
\frac{B}{3}\,h^3 - Ch + f''(x) + \frac{F}{2}\,x_1^2h = 0
$$

From this we have $F = 0$ and

$$
f''(x) \equiv L = Ch - Bh^3/3 \tag{6.140}
$$

then (6.139) is:

$$\begin{cases} \sigma_{11} &= Ax_2 + Bx_1^2x_2 - \dfrac{2B}{3}\,x_2^3 \\ \sigma_{12} &= -Bx_1x_2^2 + Cx_1 \\ \sigma_{22} &= \dfrac{B}{3}\,x_2^3 - Cx_2 + L \end{cases}$$

2. For $x_2 = -h$, $\sigma_{22} = -q$:

$$-Bh^3/3 + Ch + L = -q$$

then taking into account (6.139) we have $2L = -q$ or

$$L = -\frac{q}{2} \tag{6.141}$$

3. For $x_2 = \pm h$, $\sigma_{12} = 0$ for every x_1:

$$-Bx_1(\pm h)^2 + Cx_1 = 0$$

then $C = Bh^2$. Equations (6.140) (6.141) yield a system of two equations for B and C:

$$\begin{cases} Ch - B(h^3/3) &= L = -(q/2) \\ C &= Bh^2 \end{cases}$$

and we have: $B = -\dfrac{3q}{4h^3}$ and $C = -\dfrac{3q}{4h}$ and the relation for stress now is:

$$\begin{cases} \sigma_{11} &= Ax_2 - (3q/4h^3)x_1^2x_2 + (q/2h^3)\,x_2^3 \\ \sigma_{12} &= -(3q/4h^3)\left(h^2 - x_2^2\right)x_1 \\ \sigma_{22} &= -(3q/4h^3)\left(\dfrac{x_2^3}{3} - h^2\,x_2 + \dfrac{2h^3}{3}\right) \end{cases} \tag{6.142}$$

4. For $x = \pm l$ it is easy to check:

$$\int_{-h}^{h} \sigma_{11}\,dx_2 = 0$$

because σ_{11} is a odd function. Moreover,

$$\int_{-h}^{h} \sigma_{12}\,dx_2 = \mp ql/2$$

The last condition in (6.132) gives an equation to find A:

$$\int_{-h}^{h} \sigma_{11}\,x_2dx_2 = \int_{-h}^{h} \left[Ax_2 - (3q/4h^3)l^2x_2 + (q/2h^3)\,x_2^3\right]\,x_2dx_2 = 0$$

then we find:

$$A = \frac{3q}{4h^3}(l^2 - \frac{2}{5}h^2)$$

and finally the stress field as follows:

$$\begin{cases} \sigma_{11} &= \dfrac{3q}{4h^3}\left[(l^2 - x_1^2)x_2 + (2/3)x_2^3 - (2/5)h^2 x_2\right] \\ \sigma_{12} &= -(3q/4h^3)\left(h^2 - x_2^2\right)x_1 \\ \sigma_{22} &= -(3q/4h^3)\left(\dfrac{x_2^3}{3} - h^2\, x_2 + \dfrac{2h^3}{3}\right) \end{cases} \tag{6.143}$$

By replacing the moment of inertia $J = 2h^3/3$ in to (6.143) we rewrite it as:

$$\begin{cases} \sigma_{11} &= \dfrac{q}{2J}\left[(l^2 - x_1^2)x_2\right] + \dfrac{qx_2}{J}\left(\dfrac{x_2^2}{3} - \dfrac{h^2}{5}\right) \\ \sigma_{12} &= -\dfrac{q}{2J}\left(h^2 - x_2^2\right)x_1 \\ \sigma_{22} &= -\dfrac{q}{2J}\left(\dfrac{x_2^3}{3} - h^2\, x_2 + \dfrac{2h^3}{3}\right) \end{cases} \tag{6.144}$$

The second term in the first equation of (6.144) gives the correction for the solution of the usual theory of bending represented by the first term. Remember that by using the Saint Venant's principle the above solution is not an exact solution at the ends of the beam.

We can then find strain using the Hooke's law, then integrate (6.58) to obtain the displacement.

6.9.2 Solution using Fourier series

A function $f(x)$ defined on $(x_0; x_0 + L)$, fulfilling the following four conditions: (i) the function must be periodic; (ii) it must be single-valued and continuous, except possibly at a finite number of finite discontinuities; (iii) it must have only a finite number of maxima and minima within one period L; (iv) the integral over one period of $|f(x)|$ must converge, may be expanded as a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_n \cos\left(\frac{2m\pi x}{L}\right) + b_m \sin\left(\frac{2m\pi x}{L}\right)\right] \tag{6.145}$$

where a_0, a_m, b_m are constants called the Fourier coefficients. These coefficients are analogous to those in a power series expansion. They are given by:

$$a_m = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\,\cos\left(\frac{2m\pi x}{L}\right)dx \tag{6.146}$$

$$b_m = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\,\sin\left(\frac{2m\pi x}{L}\right)dx \tag{6.147}$$

where x_0 is arbitrary but is often taken as 0 or $-L/2$.

We can use the double Fourier series containing trigonometric functions to solve problems of narrow rectangular beams (see Figure 6.19), especially in the case of discontinuous loadings.

Consider first a stress function of the form:

$$\Phi(x_1, x_2) = f(x_2)\,\sin(\alpha x_1) \tag{6.148}$$

where $\alpha = (n\pi/l)$, n is an integer and l is the length of the beam. Substituting (6.148) into (6.114) we get a linear differential equation:

$$\frac{d^4 f}{dx_2^4} - 2\alpha^2\frac{d^2 f}{dx_2^2} + \alpha^4 f(x_2) = 0 \tag{6.149}$$

the general integral of this equation can be written in the form

$$f = (A_1 + A_2\,x_2)\exp(\alpha x_2) + (A_3 + A_4\,x_2)\exp(-\alpha x_2) \tag{6.150}$$

or in the form:

$$f = C_1\cosh(\alpha x_2) + C_2\sinh(\alpha x_2) + C_3 x_2\cosh(\alpha x_2) + C_4 x_2\sinh(\alpha x_2) \tag{6.151}$$

where the constants A_i, $(i = 1,2,3,4)$ or C_i, $(i = 1,2,3,4)$ can be found from the bounding conditions of the problem. The stress field is:

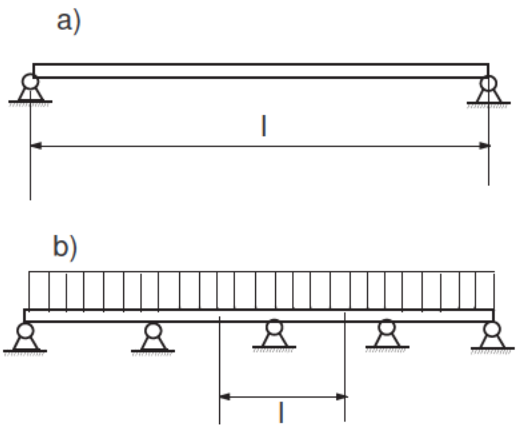

Figure 6.19:

$$\begin{aligned}
\sigma_{11} &= \frac{\partial^2\Phi}{\partial x_2^2} = \sin(\alpha x_1)\left\{C_1\alpha^2\cosh(\alpha x_2) + C_2\alpha^2\sinh(\alpha x_2) + \right.\\
&\quad \left. + C_3\alpha\left[2\sinh(\alpha x_2) + \alpha x_2\cosh(\alpha x_2)\right] + C_4\alpha\left[2\cosh(\alpha x_2) + \alpha x_2\sinh(\alpha x_2)\right]\right\}\\
\sigma_{22} &= \frac{\partial^2\Phi}{\partial x_1^2} = -\alpha^2\sin(\alpha x_1)\,\left[C_1\cosh(\alpha x_2) + C_2\sinh(\alpha x_2) + \right.\\
&\quad \left. + C_3 x_2\cosh(\alpha x_2) + C_4 x_2\sinh(\alpha x_2)\right]\\
\sigma_{12} &= -\frac{\partial^2\Phi}{\partial x_1\partial x_2} = -\alpha\cos(\alpha x_1)\left\{C_1\alpha\sinh(\alpha x_2) + C_2\alpha\cosh(\alpha x_2) + \right.\\
&\quad \left. + C_3\left[\cosh(\alpha x_2) + \alpha x_2\sinh(\alpha x_2)\right] + C_4\left[\sinh(\alpha x_2) + \alpha x_2\cosh(\alpha x_2)\right]\right\}
\end{aligned} \tag{6.152}$$

Similar solution can be obtained by supposing a stress function of the form:

$$\Phi(x_1, x_2) = f(x_2)\cos(\alpha x_1) \tag{6.153}$$

From (6.152) we see that for $x_1 = 0$ and $x_1 = l$ we have $\sigma_{11} = 0$; $\sigma_{12} \neq 0$; $u_2 = 0$; $u_1 \neq 0$ example for beams like on Figure 6.19a on roller supports. With stress function (6.153) then when $x_1 = 0$ and $x_1 = l$ we have $\sigma_{11} \neq 0$; $\sigma_{12} = 0$; $u_2 \neq 0$; $u_1 = 0$, example for part of a beam on Figure 6.19b. under symmetrical loadings.

The biggest advantage of solutions using series with trigonometric functions is that: it can be applied for problem with arbitrary distribution of loads on the upper and lower edges of a beam (see figure 6.20). In this case e.g. of normal loadings we expand the upper and lower loads as Fourier series (6.145):

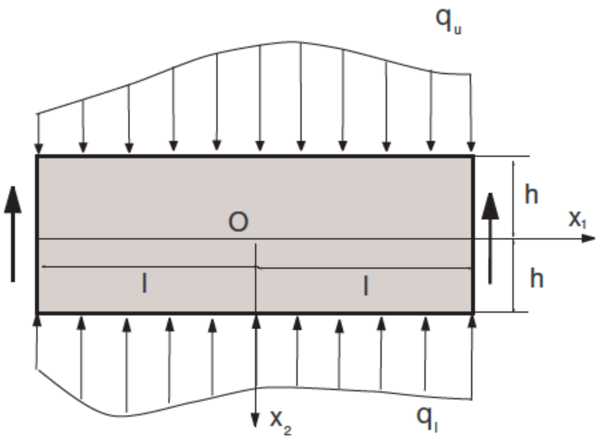

Figure 6.20:

$$q_u = A_0 + \sum_{m=1}^{\infty} \left[A_m \sin\left(\frac{m\pi x_1}{l}\right) + A'_m \cos\left(\frac{m\pi x_1}{l}\right) \right] \tag{6.154}$$

$$q_l = B_0 + \sum_{m=1}^{\infty} \left[B_m \sin\left(\frac{m\pi x_1}{l}\right) + B'_m \cos\left(\frac{m\pi x_1}{l}\right) \right] \tag{6.155}$$

where we calculate the Fourier coefficients using (6.146) and (6.147) (here $L = 2l$):

$$A_0 = \frac{1}{2l} \int_{-l}^{l} q_u(x_1) dx_1 \tag{6.156}$$

$$A_m = \frac{1}{l} \int_{-l}^{l} q_u(x_1) \sin\left(\frac{m\pi x_1}{l}\right) dx_1 \tag{6.157}$$

$$A'_m = \frac{1}{l} \int_{l}^{l} q_u(x_1) \cos\left(\frac{m\pi x_1}{l}\right) dx_1 \tag{6.158}$$

and

$$B_0 = \frac{1}{2l} \int_{-l}^{l} q_l(x_1) dx_1 \tag{6.159}$$

$$B_m = \frac{1}{l} \int_{-l}^{l} q_l(x_1) \sin\left(\frac{m\pi x_1}{l}\right) dx_1 \tag{6.160}$$

$$B'_m = \frac{1}{l} \int_{l}^{l} q_l(x_1) \cos\left(\frac{m\pi x_1}{l}\right) dx_1 \tag{6.161}$$

In such way the loads are decomposed in uniform loads A_0, B_0 which were discussed in previous section and the loads containing terms $\sin\left(\frac{m\pi x_1}{l}\right)$ and $\cos\left(\frac{m\pi x_1}{l}\right)$. The stresses produced by terms $\sin\left(\frac{m\pi x_1}{l}\right)$ are obtained by summing up the terms given by (6.152). Similarly we get the the stresses produced by the terms $\cos\left(\frac{m\pi x_1}{l}\right)$. Then we can apply the superposition principle.

In the general case of loading (see Figure 6.21), if we take only n terms, then the stress function has the form:

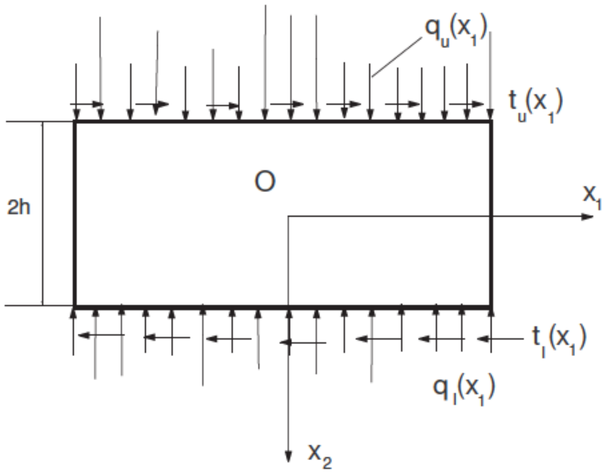

Figure 6.21:

$$\Phi = \sum_{m=1}^{n}\left[A_m\cosh\left(\frac{m\pi x_2}{l}\right)+B_m\sinh\left(\frac{m\pi x_2}{l}\right)+\right. \\ \left.+C_m x_2\cosh\left(\frac{m\pi x_2}{l}\right)+D_m x_2\sinh\left(\frac{m\pi x_2}{l}\right)\right]\sin\left(\frac{m\pi x_1}{l}\right) \tag{6.162}$$

The expression (6.162) contains $4n$ constants A_m, B_m, C_m, D_m $(i=1,2,3,4)$. We can find them using the boundary conditions:

$$\begin{aligned} \frac{\partial^2\Phi}{\partial x_1^2} &= q_u \quad ; \quad -\frac{\partial^2\Phi}{\partial x_1\partial x_2} = t_u \quad \text{for } x_2=-h \\ \frac{\partial^2\Phi}{\partial x_1^2} &= q_l \quad ; \quad -\frac{\partial^2\Phi}{\partial x_1\partial x_2} = t_l \quad \text{for } x_2=h \end{aligned} \tag{6.163}$$

As an example, consider the case (see Figure 6.22) when only:

$$q_u(x_1)=\frac{k}{l^2}(l-x_1)\,x_1 \tag{6.164}$$

Let the stress function be:

$$\Phi = \sum_{m=1}^{\infty}\left[A_m\cosh\left(\frac{m\pi x_2}{l}\right)+B_m\sinh\left(\frac{m\pi x_2}{l}\right)+\right. \\ \left.+\ C_m x_2\cosh\left(\frac{m\pi x_2}{l}\right)+D_m x_2\sinh\left(\frac{m\pi x_2}{l}\right)\right]\sin\left(\frac{m\pi x_1}{l}\right) \tag{6.165}$$

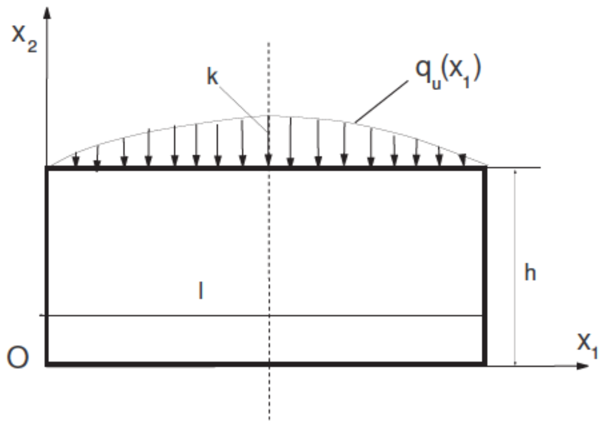

Figure 6.22:

where A_m, B_m, C_m, D_m are constants. They can be calculated directly from the conditions (6.163). We present now a general, more convenient way to calculate them from the conditions of loading on upper and lower edges.

Denote by $z = m\pi x_2/l$ as a new argument, now when $x_2 = 0$ then $z = 0$ and when $y = h$ then $z = m\beta$, with $\beta = \pi h/l$. Introduce also new functions:

$$\begin{aligned}\psi_{1m}(z) &= a_{1m}\cosh z + b_{1m}\sinh z + c_{1m}\, z\cosh z + d_{1m}\, z\sinh z\\ \psi_{2m}(z) &= a_{2m}\cosh z + b_{2m}\sinh z + c_{2m}\, z\cosh z + d_{2m}\, z\sinh z\\ \psi_{3m}(z) &= a_{3m}\cosh z + b_{3m}\sinh z + c_{3m}\, z\cosh z + d_{3m}\, z\sinh z\\ \psi_{4m}(z) &= a_{4m}\cosh z + b_{4m}\sinh z + c_{4m}\, z\cosh z + d_{4m}\, z\sinh z\end{aligned} \tag{6.166}$$

and new constants $\bar{A}_m, \bar{B}_m, \bar{C}_m, \bar{D}_m$ so that:

$$Y_m = \bar{A}_m\psi_{1m}(z) + \bar{B}_m\psi_{2m}(z) + \bar{C}_m\psi_{3m}(z) + \bar{D}_m\psi_{4m}(z) \tag{6.167}$$

now

$$\Phi = \sum_{m=1}^{\infty} Y_m(x_2)\sin\left(\frac{m\pi x_1}{l}\right)$$

The stress field is:

$$\sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2} = -\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2\sin\left(\frac{m\pi x_1}{l}\right)Y_m$$

$$\sigma_{12} = -\frac{\partial^2\Phi}{\partial x_1\partial x_2} = -\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)\cos\left(\frac{m\pi x_1}{l}\right)Y'_m$$

where $Y'_m = \dfrac{dY}{dz}\dfrac{dz}{dx_2} = \left(\dfrac{m\pi}{l}\right)\dfrac{dY}{dz}$. We have 4 boundary conditions for σ_{22} and σ_{12} on edges $x_2 = 0$ and $x_2 = h$.

We choose ψ so that the following expressions are satisfied:

$$\begin{array}{llll}\psi_{1m}(0) = 0 \;, & \psi'_{1m}(0) = 0 \;, & \psi_{1m}(m\beta) = 0 \;, & \psi'_{1m}(m\beta) = 0\\ \psi_{2m}(0) = 0 \;, & \psi'_{2m}(0) = 1 \;, & \psi_{2m}(m\beta) = 0 \;, & \psi'_{2m}(m\beta) = 0\\ \psi_{3m}(0) = 0 \;, & \psi'_{3m}(0) = 0 \;, & \psi_{3m}(m\beta) = 1 \;, & \psi'_{1m}(m\beta) = 0\\ \psi_{4m}(0) = 0 \;, & \psi'_{4m}(0) = 0 \;, & \psi_{4m}(m\beta) = 0 \;, & \psi'_{4m}(m\beta) = 1\end{array}$$

then the coefficients are given in the following table (for $k = 1, 2, 3, 4$):

	a_{km}	b_{km}	c_{km}	d_{km}
ψ_{1m}	1	$-\frac{\sinh(m\beta)\cosh(m\beta)+m\beta}{\sinh^2(m\beta)-(m\beta)^2}$	$-b_{1m}$	$-\frac{\sinh^2(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$
ψ_{2m}	0	$-\frac{(m\beta)^2}{\sinh^2(m\beta)-(m\beta)^2}$	$\frac{\sinh^2(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$	$-\frac{\sinh(m\beta)\cosh(m\beta)-m\beta}{\sinh^2(m\beta)-(m\beta)^2}$
ψ_{3m}	0	$\frac{m\beta\cosh(m\beta)+\sinh(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$	$-\frac{m\beta\cosh(m\beta)+\sinh(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$	$\frac{m\beta\sinh(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$
ψ_{4m}	0	$-\frac{m\beta\sinh(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$	$\frac{m\beta\sinh(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$	$-\frac{m\beta\cosh(m\beta)-\sinh(m\beta)}{\sinh^2(m\beta)-(m\beta)^2}$

where:

$$c_{1m} = -b_{1m};\ c_{2m} = -d_{1m};\ c_{3m} = -b_{3m};\ c_{4m} = d_{3m} = -b_{4m}$$

We see from the table that these coefficients can be calculated when we know only the ratio h/l; they are independent from the loads. The advantage of this choice of coefficients is that the constants in (6.167) have simple interpretation:

$$Y_m(0) = \bar{A}_m;\ Y'_m(0) = \bar{B}_m;\ Y_m(h) = \bar{C}_m;\ Y'_m(h) = \bar{D}_m \tag{6.168}$$

In our problem only $q_u(x_1) = \frac{k}{l^2}(l - x_1)\,x_1$, then the stress Airy's function is:

$$\Phi(x_1, x_2) = -\sum_{m=1}^{\infty}\left(\frac{l}{m\pi}\right)^2 C_m\,\psi_m(z)\sin\frac{m\pi x_1}{l} \qquad \left(z = \frac{m\pi x_2}{l}\right) \tag{6.169}$$

with

$$\psi_m(z) = b_m\sinh z + c_m z\cosh z + d_m z\sinh z \tag{6.170}$$

In (6.170) the coefficients b_m, c_m, d_m are independent from the form of loads and equal to:

$$\begin{aligned}
b_m &= \frac{m\beta\cosh m\beta + \sinh m\beta}{(\sinh m\beta)^2 - (m\beta)^2}\\
c_m &= -\frac{m\beta\cosh m\beta + \sinh m\beta}{(\sinh m\beta)^2 - (m\beta)^2}\\
d_m &= \frac{m\beta\sinh m\beta}{(\sinh m\beta)^2 - (m\beta)^2}\\
\beta &= \frac{\pi h}{l}
\end{aligned}$$

The constants C_m we can find from the expansion of function $q_u(x_1)$ in Fourier series. Using (6.145)-(6.147) we find

$$C_m = -\frac{2}{l}\int_0^l q_u(x_1)\,\sin\frac{m\pi x_1}{l}dx \tag{6.171}$$

and we have:

$$C_m = -\frac{32k}{(m\pi^3)} \qquad \text{for m } \ 1,3,5... \tag{6.172}$$

Hence:

$$\begin{aligned}
\sigma_{11} &= -\sum_{m=1,3,5,...}^{\infty} C_m\psi''_m(z)\,\sin\frac{m\pi x_1}{l}\\
\sigma_{22} &= \sum_{m=1,3,5,...}^{\infty} C_m\psi_m(z)\,\sin\frac{m\pi x_1}{l}\\
\sigma_{12} &= \sum_{m=1,3,5,...}^{\infty} C_m\psi'_m(z)\,\cos\frac{m\pi x_1}{l}
\end{aligned} \tag{6.173}$$

where ψ'_m and ψ''_m are derivatives of the first and second order of the function $\psi_m(z)$ in (6.170) in z:

$$\psi'_m(z) = d_m \sinh z + d_m z \cosh z + c_m z \sinh z \tag{6.174}$$

and

$$\psi''_m(z) = 2d_m \cosh z + c_m \sinh z + c_m z \cosh z + d_m z \sinh z \tag{6.175}$$

Having the stress field, by using the Hooke's law, we obtain:

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\sigma_{22}\right] = -\frac{1}{E}\sum_{m=1,3,5,\ldots}^{\infty} C_m \left[\psi''_m(z) + \nu\psi_m(z)\right] \sin\frac{m\pi x_1}{l} \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{11}\right] = \frac{1}{E}\sum_{m=1,3,5,\ldots}^{\infty} C_m \left[\psi_m(z) + \nu\psi''_m(z)\right] \sin\frac{m\pi x_1}{l} \\
\varepsilon_{12} &= \frac{1+\nu}{E}\,\sigma_{12} = \frac{1+\nu}{E}\sum_{m=1,3,5,\ldots}^{\infty} C_m \psi'_m(z) \cos\frac{m\pi x_1}{l}
\end{aligned} \tag{6.176}$$

By integrating relations (6.176) and using the boundary conditions, we have the displacement:

$$\begin{aligned}
u_1 &= \frac{l}{E\pi}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m}\, C_m \left[\psi''_m(z) + \nu\psi_m(z)\right] \cos\frac{m\pi x_1}{l} \\
u_2 &= \frac{l}{E\pi}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m}\, C_m \left[\Psi_m(z) + \nu\psi'_m(z)\right] \sin\frac{m\pi x_1}{l}
\end{aligned} \tag{6.177}$$

where

$$\Psi_m(z) = (b_m - c_m)\cosh z - d_m \sinh z + d_m z \cosh z + c_m z \sinh z \tag{6.178}$$

thus the problem is solved.

6.10 Solution in polar coordinates

Sometimes, for example in discussing the stress state in circular rings or disks, it is more convenient to use polar coordinates (see Figure 6.23). The relations between Cartesian and polar coordinates are:

$$x_1 = r\cos\vartheta\,,\; x_2 = r\sin\vartheta \;\text{ or }\; r^2 = x_1^2 + x_2^2\,,\; \vartheta = \arctan\frac{x_2}{x_1} \tag{6.179}$$

From (6.179) we have:

$$\begin{gathered}
\frac{\partial r}{\partial x_1} = \frac{x_1}{r} = \cos\vartheta\;;\; \frac{\partial r}{\partial x_2} = \frac{x_2}{r} = \sin\vartheta \\
\frac{\partial\vartheta}{\partial x_1} = -\frac{x_2}{r^2} = -\frac{\sin\vartheta}{r}\;;\; \frac{\partial\vartheta}{\partial x_2} = \frac{x_1}{r^2} = \frac{\cos\vartheta}{r} \\
\frac{\partial\Phi}{\partial x_1} = \frac{\partial\Phi}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial\Phi}{\partial\vartheta}\frac{\partial\vartheta}{\partial x_1} = \frac{\partial\Phi}{\partial r}\cos\vartheta - \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\sin\vartheta
\end{gathered}$$

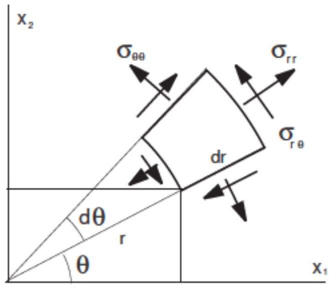

Figure 6.23: The polar coordinates system

$$\frac{\partial^2\Phi}{\partial x_1^2} = \left(\frac{\partial}{\partial r}\cos\vartheta - \frac{1}{r}\frac{\partial}{\partial\vartheta}\sin\vartheta\right)\left(\frac{\partial\Phi}{\partial r}\cos\vartheta - \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\sin\vartheta\right) =$$

$$= \frac{\partial^2\Phi}{\partial r^2}\cos^2\vartheta - \frac{2}{r}\frac{\partial^2\Phi}{\partial r\partial\vartheta}\sin\vartheta\cos\vartheta + \frac{1}{r}\frac{\partial\Phi}{\partial r}\sin^2\vartheta + \frac{2}{r^2}\frac{\partial\Phi}{\partial\vartheta}\sin\vartheta\cos\vartheta + \frac{\partial^2\Phi}{\partial\vartheta^2}\frac{\sin^2\vartheta}{r^2}$$

$$\frac{\partial^2\Phi}{\partial x_2^2} = \frac{\partial^2\Phi}{\partial r^2}\sin^2\vartheta + \frac{2}{r}\frac{\partial^2\Phi}{\partial r\partial\vartheta}\sin\vartheta\cos\vartheta + \frac{1}{r}\frac{\partial\Phi}{\partial r}\cos^2\vartheta - \frac{2}{r^2}\frac{\partial\Phi}{\partial\vartheta}\sin\vartheta\cos\vartheta + \frac{\partial^2\Phi}{\partial\vartheta^2}\frac{\cos^2\vartheta}{r^2}$$

By adding the last two expressions we obtain the Laplace operator in polar coordinates:

$$\frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial x_2^2} = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\vartheta^2}$$

The biharmonic equation $\Delta\Delta\Phi = 0$ now takes the form:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}\right)\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\vartheta^2}\right) = 0 \tag{6.180}$$

Then we can calculate the components of stress in polar coordinates:

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\vartheta^2} \\ \sigma_{r\vartheta} &= \frac{1}{r^2}\frac{\partial\Phi}{\partial\vartheta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\vartheta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\right) \\ \sigma_{\vartheta\vartheta} &= \frac{\partial^2\Phi}{\partial r^2} \end{aligned} \tag{6.181}$$

The equilibrium equations can be obtained by considering the equilibrium of a small element on Figure 6.23, and take the forms:

$$\begin{aligned} \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\vartheta}}{\partial\vartheta} + \frac{\sigma_{rr} - \sigma_{\vartheta\vartheta}}{r} &= 0 \\ \frac{1}{r}\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_{r\vartheta}}{\partial r} + \frac{2\sigma_{r\vartheta}}{r} &= 0 \end{aligned} \tag{6.182}$$

When the stress distribution is symmetrical with respect to the axis through O and perpendicular to the plane x_1Ox_2, then the shear stress $\sigma_{r\vartheta} = 0$ (see the second equation of (6.181)), and other components of stress are functions of r only and do not depend on ϑ, the partial derivative is the ordinary derivative, then:

$$\Delta\Delta\Phi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr}\right) = \frac{d^4\Phi}{dr^4} + \frac{2}{r}\frac{d^3\Phi}{dr^3} - \frac{1}{r^2}\frac{d^2\Phi}{dr^2} + \frac{1}{r^3}\frac{d\Phi}{dr} = 0 \quad (6.183)$$

This is an ordinary differential equation. Let's find the solution in the form:

$$\Phi = r^n \quad (6.184)$$

By substituting this into (6.183) we get the following equation for n:

$$n(n-1)(n-2)(n-3) + 2n(n-1)(n-2) - n(n-1) + n = 0 \quad (6.185)$$

with the roots: $n_1 = 0, n_2 = 2$, $n_3 = n_4 = 1$, the general integral of equation (6.183) is:

$$\Phi = A\ln r + Br^2\ln r + Cr^2 + D \quad (6.186)$$

From this we calculate the stress components following (6.181):

$$\begin{aligned} \sigma_{rr} &= \frac{A}{r^2} + B(1 + 2\ln r) + 2C \\ \sigma_{r\vartheta} &= 0 \\ \sigma_{\vartheta\vartheta} &= -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \end{aligned} \quad (6.187)$$

If there is no hole at the origin O the constants A and B must be vanish, since otherwise the components of stress become infinite when $r \to 0$. If there is a hole at $r = 0$, the study of displacement shows that $B = 0$, then the stress components now are:

$$\sigma_{rr} = \frac{A}{r^2} + 2C \ ; \ \sigma_{\vartheta\vartheta} = -\frac{A}{r^2} + 2C \quad (6.188)$$

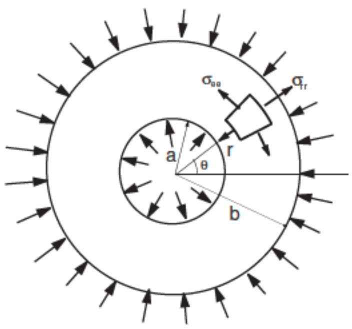

Figure 6.24:

Problem 19 *Find the stress field in a hollow cylinder of inner radius a and outer radius b is subjected to an internal pressure p_i and an external pressure p_o.*

Solution:

The boundary conditions are:

$$\sigma_{rr}|_{r=a} = -p_i \; ; \; \sigma_{rr}|_{r=b} = -p_o \tag{6.189}$$

By substituting (6.189) into the first equation of (6.188) we can find the constants A and C, then finally we have the stress field:

$$\begin{aligned} \sigma_{rr} &= \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \\ \sigma_{\vartheta\vartheta} &= -\frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \end{aligned} \tag{6.190}$$

In a particular case, when $p_o = 0$, the cylinder is subjected only to an internal pressure, from (6.190) we have:

$$\begin{aligned} \sigma_{rr} &= \frac{p_i a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \\ \sigma_{\vartheta\vartheta} &= \frac{p_i a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \end{aligned} \tag{6.191}$$

From this relation we see that the stress $\sigma_{\vartheta\vartheta}$ is always a tensile stress and gets the maximum value at the inner surface of the cylinder. For thin-walled cylinder, $b \to a$, denoting $b - a = t$ then we have $\sigma_{\vartheta\vartheta} = p_i a/t$.

Chapter 7

Plasticity

7.1 One dimensional models

The characteristic stages of material behaviour for a ductile material can be illustrated by a typical stress-strain (or loading-elongation) relationship as shown in Figure 7.1 in a tension test. This highly nonlinear relationship can be roughly divided into five intervals [5].

Within the linear portion OA (called the proportional range), if the load is reduced to zero (i.e., unloading), then the line OA is retraced back to O and the specimen has exhibited an *elasticity*. Applying a load that is greater than A and then unloading, we typically traverse $OABH$ and find that there is a "permanent elongation" OH. Reapplication of the load from H indicates elastic behavior with the same slope as OA but with an increased proportional limit. The material is said to have *work-hardening*. The unloading on portion BH has the same slope. Hence, the plastic deformation does not affect elastic properties of the material, so that the unloading slope (Young's modulus E) remains the same as before the plastic deformation took place. At an arbitrary stage of this elastic-plastic deformation, the total strain is the sum of elastic (which still obeys Hooke's law) and plastic parts ($\varepsilon = \varepsilon^e + \varepsilon^p$). Next, we have a portion where the stress is constant and the strain continually grows, the material has exhibited a *perfect plasticity*. After a maximum of the stress strain curve, deformation localizes to form a neck.

Mathematical descriptions of complicated true stress-strain curves are thus needed. With elastic deformation, the strains are proportional to the stress. This model is described by the Hooke's law presented in Chapter 6 (see Figure 7.2a):

$$\sigma = E\,\varepsilon \tag{7.1}$$

When the stress is increased beyond the initial yield limit Y, the material deforms plastically. By neglecting the hardening interval in Figure 7.1, we have a model elastic perfect plastic material presented in Fig. 7.2b:

$$\sigma = \begin{cases} E\,\varepsilon & \text{for} \quad \varepsilon \le \varepsilon_Y \\ Y & \text{for} \quad \varepsilon > \varepsilon_Y \end{cases} \tag{7.2}$$

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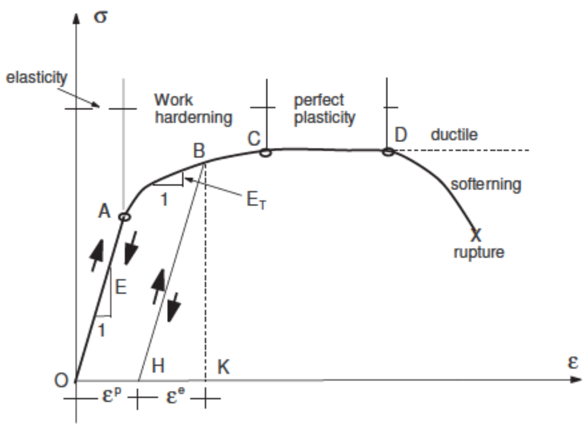

Figure 7.1:

Because the elastic part of the strain is usually much less than the plastic part, when it will be neglected and we have a model of rigid perfect plastic material (see Figure 7.2d):

$$\sigma = Y \tag{7.3}$$

We see that a definite level of stress Y must be applied before any plastic deformation occurs. As the stress is further increased, the amount of deformation increases, but in general, not linearly. A simpler model for linear work-hardening is shown on Fig. 7.2c. After plastic deformation starts, the total strain is the sum of the elastic strain and the plastic strain. We denote next yield level by σ_Y:

$$\begin{aligned} \sigma &= E\,\varepsilon && \text{for} \quad \varepsilon \leq \varepsilon_Y \\ \sigma - Y &= E_1\,(\varepsilon - \varepsilon_Y) && \text{for} \quad \varepsilon > \varepsilon_Y \end{aligned} \tag{7.4}$$

7.2 Rheological Models

In classic elasticity, we assume that there is no time delay between application of a force and the deformation that it causes. However, for many materials there is additional time-dependent deformation that is recoverable. This is called *viscoelastic* or *anelastic* deformation. When a load is applied to a material, there is an instantaneous elastic response but the deformation also increases with time. Anelastic strains in metals and ceramics are usually small and are ignored, but in many polymers viscoelastic strains can be very significant. During the deformation process, elastic materials store all energy obtained, while anelastic material loses a part of energy because of dissipation. Its stress-strain curve has a hysteresis loop. In practice, this kind of material can be used to make a cover to prevent shock on goods. The other properties are *relaxation* and *creep* (see Figure 7.3). Relaxation is the change of stress (unloading) when strain is constant. Creep is the strain response under constant stress. This may be a very slow process, with very small velocity and can last even few months.

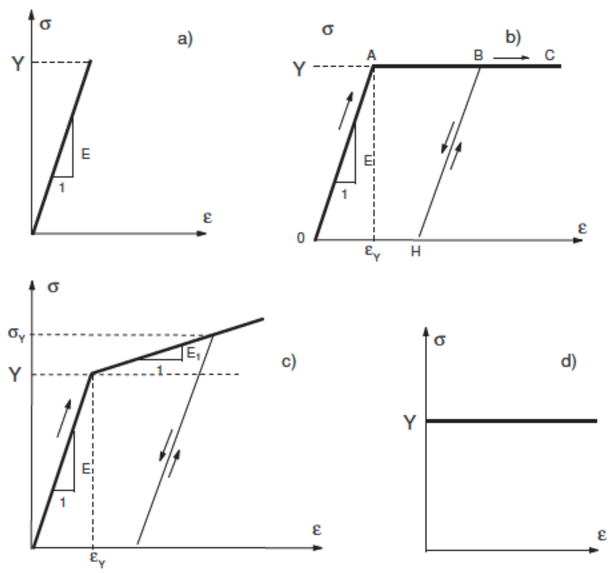

Figure 7.2: Mathematical approximations of the true stress-strain curve: a)Elasticity; b) Elastic-Perfect Plasticity; c) Elastic-Plasticity with Linear Hardening; d) Rigid Perfect Plasticity

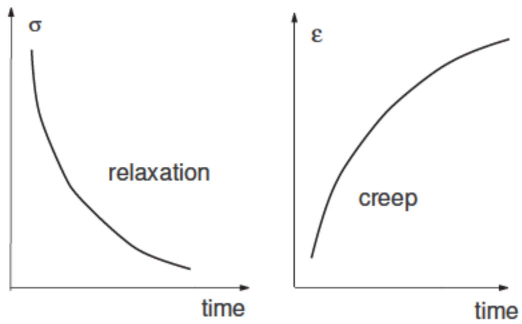

Figure 7.3:

Anelastic behavior can be modeled mathematically with structures constructed from idealized elements representing elastic and viscous behavior. In these models the stress σ is replaced by the force Q, and the strain ε - by the displacement u of the applied point. Dividing the force by the area, the displacement by a length, we have the relation $\sigma = f(\varepsilon)$.

A spring models a perfectly elastic solid (see Figure 7.4), the behavior of which is described by $\sigma = E\,\varepsilon$, where E is the elastic spring constant. A dashpot models a perfectly viscous material. This model is related to the Newton's law of viscous liquids. Its behavior is described by $\sigma = \eta\dot{\varepsilon}$ where η is the viscous dashpot constant, and the superposed dot indicates time derivative. It is understood that the spring element responds instantly to a stress, while the dashpot cannot respond instantly (because its response is rate dependent).

Series combination of a spring and a dashpot gives us *the Maxwell model*. When elements are connected in series, each element carries the same amount of stress while the

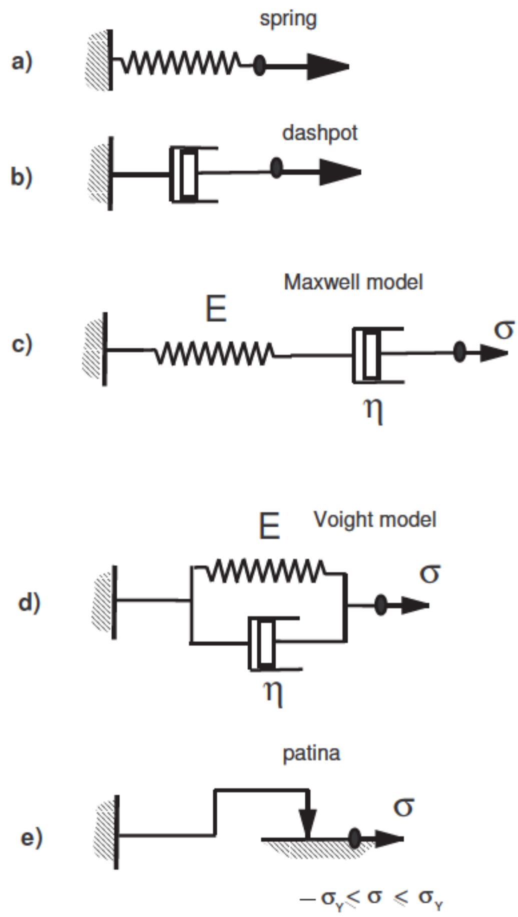

Figure 7.4:

strains are different in each element. We have:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{7.5}$$

or

$$\sigma + \frac{\eta}{E}\frac{d\sigma}{dt} = \eta \frac{d\varepsilon}{dt} \tag{7.6}$$

Consider the response of this model when $\varepsilon = const$ and $\sigma = const$. We start with a test $\varepsilon = \varepsilon_0 = const$, then $\dot{\varepsilon} = 0$ and we obtain from (7.5):

$$0 = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

After integrating, we get:

$$\sigma = C \exp\left(-\frac{E}{\eta}t\right)$$

where C - is an integration constant. To find it integrate the equation (7.5) from $t = 0^-$ to a very small value $t = \tau$:

$$\varepsilon(\tau) - \varepsilon(0^-) = \frac{1}{E}[\sigma(\tau) - \sigma(0^-)] + \frac{1}{\eta}\sigma_{mean}.\tau$$

Because of $\varepsilon(0^-) = \sigma(0^-) = 0$, $\varepsilon(\tau) = \varepsilon_0$, $\sigma(\tau) = \sigma_0$, so $C = \sigma_0 = E\varepsilon_0$, then:

$$\sigma = E\varepsilon_0 \exp\left(-\frac{E}{\eta}t\right) \tag{7.7}$$

Dividing (7.7) by ε_0 we get the so-called *relaxation function*:

$$R(t) = E \exp\left(-\frac{E}{\eta}t\right) \tag{7.8}$$

The time $t_R = \eta/E$ is known as the *relaxation time*. In this model, when $t \to \infty$, $\sigma = 0$.

Consider the test $\sigma = \sigma_0 = const$. From (7.5) we can get:

$$\varepsilon(t) = \left(\frac{1}{E}t + \frac{1}{\eta}\right)\sigma_0 \tag{7.9}$$

Similar as in (7.8), dividing relation (7.9) by σ_0, we have the *creep function*:

$$P(t) = \left(\frac{1}{E}t + \frac{1}{\eta}\right) \tag{7.10}$$

The relaxation and creep curves for this model are shown in Figure 7.5. The relaxation

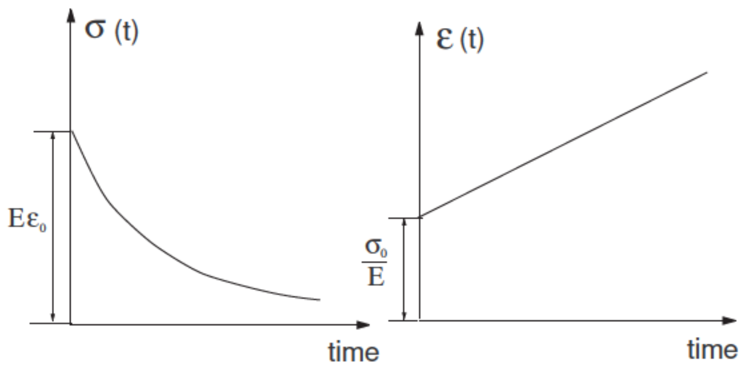

Figure 7.5:

here is similar to that of the fluid Newton, while the creep does not complies with the experiments.

The Kelvin-Voight element of Figure 7.4 consists of a linear elastic spring element in parallel with a dashpot element. Each element carries now the same amount of strain.

Let σ_1 be the stress in the spring and σ_2 be the stress in the dashpot. Then:

$$\sigma(t) = \sigma_1 + \sigma_2 = E\varepsilon(t) + \eta\dot{\varepsilon}(t) \tag{7.11}$$

In the same way, we find now the relaxation and the creep function for this model. With the test $\varepsilon = const$ by integrating (7.11) from 0^- to τ

$$\int_{0-}^{\tau} \sigma dt = \int_{0-}^{\tau} E\varepsilon dt + \int_{0-}^{\tau} \eta\dot{\varepsilon} dt$$

where

$$\int_{0-}^{\tau} \eta\dot{\varepsilon} dt = \eta[\varepsilon(\tau) - \varepsilon(0^-)] = \eta\varepsilon_0$$

$$\int_{0^-}^{\tau} E\varepsilon dt = 0$$

From the geometrical interpretation of integral, the first integral is the area under the curve $\sigma(t)$. It must have a finite value, otherwise the stress takes infinite value. By introducing the Dirac function, which has the following property:

$$\int_{0^-}^{\tau} \delta(t)dt = 1 \tag{7.12}$$

then the initial condition can be written in the form:

$$\sigma(t) = \eta\varepsilon_0\delta(t) \tag{7.13}$$

then:

$$\sigma(t) = E\varepsilon_0 + \eta\varepsilon_0\delta(t) \tag{7.14}$$

and the relaxation function is:

$$R(t) = E + \eta\,\delta(t) \tag{7.15}$$

that means with the presence of the dashpot, we get a relaxation without a delay .

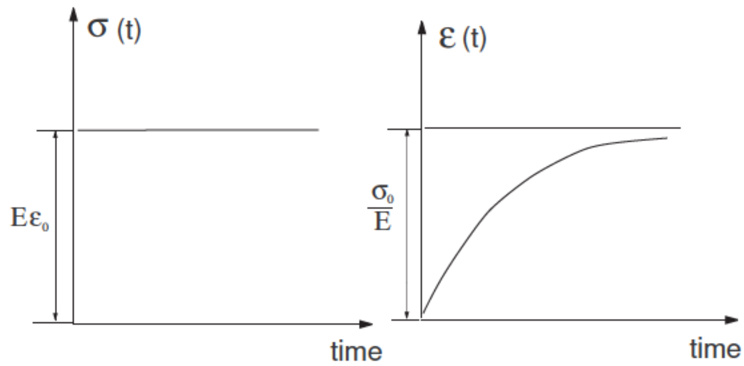

Figure 7.6:

Now for the test of constant stress $\sigma = \sigma_0$, we have the differential equation:

$$\frac{\sigma_0}{\eta} = \frac{E}{\eta}\varepsilon(t) + \dot{\varepsilon}(t) \tag{7.16}$$

By integrating, we obtain:

$$\varepsilon(t) = \frac{\sigma_0}{E} + C\exp\left(-\frac{E}{\eta}t\right)$$

For $t = 0$, $\varepsilon = \varepsilon_0$, then $C = -\sigma_0/E$ and

$$\varepsilon(t) = \frac{\sigma_0}{E}\left[1 - \exp\left(-\frac{E}{\eta}t\right)\right] \tag{7.17}$$

so the creep function is:

$$P(t) = \frac{1}{E}\left[1 - \exp\left(-\frac{E}{\eta}t\right)\right] \tag{7.18}$$

The relaxation and creep curves for the model Kenvil-Voight are shown in Figure 7.6.

An elementary mechanical system describing the behaviour of plasticity is the patina (see Figure 7.4e), which describe the appearance of the permanent deformation if the loading is big enough, using Coulomb law of friction. If the first step of the permanent deformation does not evaluate during the loading, the behaviour is perfect plastic and moreover, if the deformation between the flow is neglected, the model is rigid-perfect plastic.

The association between a spring and a patina in series produces a elastic perfect plastic behaviour (fig. 7.7) the system not being able to support a stress which's absolute value is bigger then σ_Y.

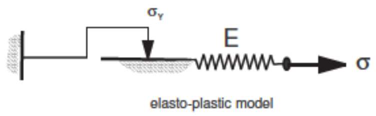

Figure 7.7:

The spring, dashpot and patina can be combined, making different rheological models. In the first two sections of this chapter, we have obtained the stress-strain relations for the one-dimensional case. We go now to the formulation of constitutive equations that describe elastic-plastic material response under multiaxial states of stress, similarly to how from equation (6.1) we get the Hooke's law of the form (6.22) for isotropic solids.

7.3 Elastic-Perfect Plastic Materials

From the first section we know that material is elastic until it reaches the yield limit Y. When the stress is over such value, then the total strain in a plastic material can be considered as the sum of the reversible elastic strain and the permanent plastic strain. If the material does not admit changes of the permanent strain under constant stress, then the material is called *perfectly plastic*, otherwise the material is called *work-hardening*. In this section we study the elastic-perfect plastic material, where we assume that the yield strengths in tension and compression are equal, which is consistent with the observation for most materials. It means that we neglect in this section the Bauschinger effect, when the compressive yield strengths are generally greater than the tensile yield strengths e.g. for polymers. Next, we introduce a criterion for *loading*: from the point B on the stress-strain curve for elastic-perfect plastic material on Figure 7.2, we can continue loading (go to the point C) or have an *unloading*, when material behaviour is elastic. We go now to the details.

7.3.1 Criteria of loading and unloading

1. For elastic perfect plastic material, the behaviour is elastic until it reaches the yield limit (Figure 7.1). A yield criterion is a mathematical expression of the stress states

that will cause yielding or plastic flow

$$f(\sigma_{ij}) = k, \ (k = const) \tag{7.19}$$

2. Then plastic deformation takes place. For the plastic flow to continue, the state of stress must remain on yield surface. This is called the criterion for loading:

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \tag{7.20}$$

3. When stresses are removed or where the stress intensity drops below the yield value, we have the unloading:

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \tag{7.21}$$

In nine-dimensional stress space, the equation (7.19) $f(\sigma_{ij}) = 0$ represents a hypersurface (see figure 7.8): For the elastic-perfect plastic material, the yield function is a

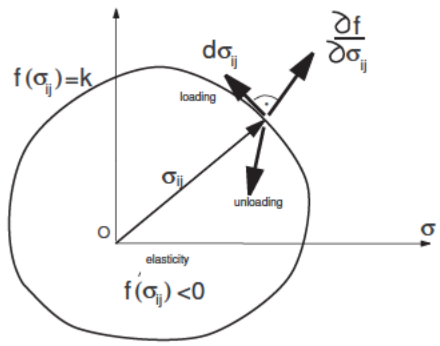

Figure 7.8:

fixed surface in stress space. Each point inside the surface represents an elastic state and each point on the surface represents a plastic state.

During loading, both elastic and elastic strains occur, then the total strain increment is the sum of of elastic and plastic parts:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p} \tag{7.22}$$

while during unloading, the material behaviour is only elastic, then:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} \tag{7.23}$$

The increment of elastic strain satisfies the Hooke's law (see (6.32)):

$$d\varepsilon_{ij}^{e} = \frac{1}{E}\left[(1+\nu)\, d\sigma_{ij} - \nu\, d\sigma_{kk}\, \delta_{ij}\right] = \frac{1}{E}\left[(1+\nu)\, d\sigma_{ij} - \nu\, dI_\sigma\, \delta_{ij}\right] \tag{7.24}$$

where $I_\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33}$ is the first invariant of the stress tensor σ_{ij}. It will be shown later that plastic deformation causes no volume change, so it is more convenient to use the deviator of stress in relations between stress and strain. Remember the decomposition of this tensor (see (4.30)):

$$\sigma_{ij} = s_{ij} + \frac{1}{3} I_\sigma\, \delta_{ij} \tag{7.25}$$

where s_{ij} is the deviator of the stress tensor σ, then:

$$d\sigma_{ij} = ds_{ij} + \frac{1}{3}\, dI_\sigma\, \delta_{ij} \tag{7.26}$$

we can write the increment of elastic strain in the form:

$$d\varepsilon^e_{ij} = \frac{ds_{ij}}{2G} + \frac{dI_\sigma}{9K}\, \delta_{ij} \tag{7.27}$$

where $G = E/2(1+\nu)$ is the elastic shear modulus and $K = E/3(1-2\nu)$ is the bulk modulus (see table 6.2). Introducing the concept of a *plastic potential function* $g(\sigma_{ij})$, which enables us to write the increment of the plastic flow in the form:

$$d\varepsilon^p_{ij} = d\lambda\, \frac{\partial g}{\partial \sigma_{ij}} \tag{7.28}$$

where $d\lambda$ is a positive scalar factor of proportionality, which is non-zero only when plastic deformations occur. In the special case, when the plastic potential function and the yield function coincide, $f = g$, we have:

$$d\varepsilon^p_{ij} = d\lambda\, \frac{\partial f}{\partial \sigma_{ij}} \tag{7.29}$$

The relation (7.29) is called then *associated flow rule* because this is connected with the yield function. Relation (7.28) with $g \neq f$ is called the *non-associated flow rule*. Using (2.52) we can conclude that the plastic increment $d\varepsilon^p_{ij}$ has the direction of the normal vector to the yield surface (see Figure 7.8).

7.3.2 Yield Functions

A yield criterion is a mathematical expression of the stress states that will cause yielding or plastic flow. The most general form of a yield criterion is

$$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}) = f(\sigma_{ij}) = C \tag{7.30}$$

where C is a material constant. For an isotropic material this can be expressed in terms of principal stresses:

$$f(\sigma_1, \sigma_2, \sigma_3) = C \tag{7.31}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses, the roots of equation (4.25).

The simplest yield criterion is proposed by Tresca. It states that yielding will occur when the largest shear stress reaches a critical value. The largest shear stress is $\tau_{max} = (\sigma_{max} - \sigma_{min})/2$ (see (4.39)), so the Tresca criterion can be expressed as

$$\sigma_{max} - \sigma_{min} = C \tag{7.32}$$

Taking the convention that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the equation (7.32) can be written as:

$$\sigma_1 - \sigma_3 = C \tag{7.33}$$

The constant C can be found by considering uniaxial tension. In this state $\sigma_1 = Y$, $\sigma_2 = \sigma_3 = 0$, then $C = Y$, Y is the tension yield strength, and the Tresca yield function is:

$$\sigma_1 - \sigma_3 = Y \tag{7.34}$$

For a state of pure shear $\sigma_1 = -\sigma_3 = k$, $\sigma_2 = 0$ equation (7.34) gives $2k = Y$ so:

$$\sigma_1 - \sigma_3 = Y = 2k \tag{7.35}$$

where k is shear yield strength.

In Tresca yield function the effect of the intermediate principal stress σ_2 is neglected. This is included in the Huber-Mises criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 \tag{7.36}$$

For a state of pure shear $\sigma_1 = -\sigma_3 = k$, $\sigma_2 = 0$ equation (7.36) gives: $(k)^2 + (0-k)^2 + (-k-k)^2 = 2Y^2$, then:

$$k = \frac{Y}{\sqrt{3}} \tag{7.37}$$

Taking into account (4.34) and (4.36), we can also write:

$$II_s = \frac{1}{2} s_{ij}\, s_{ij} = \frac{Y}{\sqrt{3}} = k \tag{7.38}$$

where k means the yield stress in simple shear.

In case of plane stress, when $\sigma_3 = 0$, the criterion (7.9) is:

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\,\sigma_2 = Y^2 \tag{7.39}$$

which is an ellipse in plane σ_1, σ_2 as in Figure: It can be shown that in this plane, the

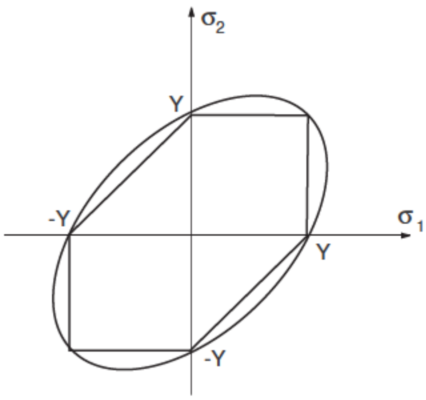

Figure 7.9:

Tresca criterion is the hexagon inscribe in that ellipse.

This criterion can also be expressed in terms of stresses that are not principal stresses in the form:

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\,(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 2Y^2 \tag{7.40}$$

The relation (7.40) in the case of plane stress is:

$$\sigma_{11}^2 - \sigma_{11}\,\sigma_{22} + \sigma_{22}^2 + 3\,\sigma_{12}^2 = Y^2 \tag{7.41}$$

7.3.3 Incremental stress-strain relation

In view of (7.27) and (7.29) we have the complete stress-strain relation for an elastic-perfect plastic material [4]:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p} = \frac{ds_{ij}}{2G} + \frac{dI_\sigma}{9K}\,\delta_{ij} + d\lambda\,\frac{\partial f}{\partial \sigma_{ij}} \tag{7.42}$$

where $d\lambda$ is still unknown factor. In view of (7.19), (7.20) and (7.21), we have:

$$d\lambda \begin{cases} > 0 & \text{whenever} \quad f = 0 \quad \text{and} \quad df = 0 \\ = 0 & \text{whenever} \quad f < 0 \quad \text{or} \quad f = 0 \quad \text{but } df < 0 \end{cases} \tag{7.43}$$

The relation $df = 0$ can be written as follows:

$$df = \frac{\partial f}{\partial \sigma_{ij}}\, d\,\sigma_{ij} = 0 \tag{7.44}$$

and is known as the *consistency condition*. Substituting ds_{ij} from (7.42) into (7.26) we have:

$$d\sigma_{ij} = 2G d\varepsilon_{ij} - \frac{2G}{9K} dI_\sigma\,\delta_{ij} - 2G\,d\lambda \frac{\partial f}{\partial \sigma_{ij}} + \frac{1}{3}\,dI_\sigma\,\delta_{ij} \tag{7.45}$$

Substituting (7.45) into (7.44):

$$2G\,\frac{\partial f}{\partial \sigma_{ij}}\,d\varepsilon_{ij} - 2G\,d\lambda\,\frac{\partial f}{\partial \sigma_{ij}}\,\frac{\partial f}{\partial \sigma_{ij}} + \left(\frac{1}{3} - \frac{2G}{9K}\right) dI_\sigma\,\frac{\partial f}{\partial \sigma_{ij}}\,\delta_{ij} = 0 \tag{7.46}$$

With $i = j$, relation (7.42) gives:

$$dI_\sigma = 3K\left(d\varepsilon_{ij} - d\lambda\,\frac{\partial f}{\partial \sigma_{ij}}\delta_{ij}\right) \tag{7.47}$$

Substituting (7.47) into (7.46), we get a relation for $d\lambda$:

$$d\lambda = \frac{\dfrac{\partial f}{\partial \sigma_{ij}}\,d\varepsilon_{ij} + \dfrac{3K - 2G}{6G}\,d\varepsilon_{kk}\left(\dfrac{\partial f}{\partial \sigma_{ij}}\,\delta_{ij}\right)}{\dfrac{\partial f}{\partial \sigma_{ij}}\,\dfrac{\partial f}{\partial \sigma_{ij}} + \dfrac{3K - 2G}{6G}\left(\dfrac{\partial f}{\partial \sigma_{ij}}\,\delta_{ij}\right)^2} \tag{7.48}$$

In the case of The Huber-Mises criterion (see (7.38)):

$$f = \sqrt{II_s} = \sqrt{\frac{1}{2}\,s_{ij}\,s_{ij}} = k \tag{7.49}$$

We have:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial \sqrt{II_s}}{\partial \sigma_{ij}} = \frac{1}{2\sqrt{II_s}}\,\frac{\partial \sqrt{II_s}}{\partial \sigma_{ij}} = \frac{1}{2\sqrt{II_s}}\,s_{kl}\,\frac{\partial s_{kl}}{\partial \sigma_{ij}} \tag{7.50}$$

Since $s_{kl} = \sigma_{kl} - (1/3)\,\sigma_{mm}\,\delta_{kl}$ (see (4.30)), then:

$$\frac{\partial s_{kl}}{\partial \sigma_{ij}} = \frac{1}{2}\,(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) + \frac{1}{3}\,\delta_{mi}\delta_{mj}\,\delta_{kl} = \frac{\partial s_{kl}}{\partial \sigma_{ij}} = \frac{1}{2}\,(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) + \frac{1}{3}\,\delta_{ij}\,\delta_{kl} \tag{7.51}$$

Substituting (7.51) into (7.50), one gets:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{1}{2\sqrt{II_s}}\,s_{kl}\left[\frac{1}{2}\,(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) + \frac{1}{3}\,\delta_{ij}\,\delta_{kl}\right] = \frac{1}{2\sqrt{II_s}}\,s_{ij} \tag{7.52}$$

Then since $\dfrac{\partial f}{\partial \sigma_{ij}}\,\delta_{ij} = \dfrac{1}{2\sqrt{II_s}}\,s_{ij}\,\delta_{ij} = \dfrac{1}{2\sqrt{II_s}}\,s_{ii} = 0$, from (7.48) we obtain:

$$d\lambda = \frac{\dfrac{\partial f}{\partial \sigma_{ij}}\,d\varepsilon_{ij}}{\dfrac{\partial f}{\partial \sigma_{ij}}\,\dfrac{\partial f}{\partial \sigma_{ij}}} = \frac{\dfrac{s_{ij}\,d\varepsilon_{ij}}{2\sqrt{II_s}}}{\dfrac{s_{ij}}{2\sqrt{II_s}}\,\dfrac{s_{ij}}{2\sqrt{II_s}}} = \frac{s_{ij}\,d\varepsilon_{ij}}{k} = \frac{s_{ij}\,de_{ij}}{k} \tag{7.53}$$

Here de_{ij} is the increment of the deviatoric part of strain tensor, $s_{ij}\,d\varepsilon_{ij} = s_{ij}\,[de_{ij} + (1/3)\,d\varepsilon_{kk}]\,\delta_{ij} = s_{ij}\,de_{ij}$ because of the first invariant of deviator of stress tensor $s_{ij}\,\delta_{ij} = s_{kk} = 0$.

Substituting $d\lambda$ (7.53) into (7.42), (7.45), we get the complete stress-strain relations:

$$\begin{cases} d\varepsilon_{ij} &= \dfrac{ds_{ij}}{2G} + \dfrac{dI_\sigma}{9K}\,\delta_{ij} + \dfrac{s_{mn}\,de_{mn}}{2k^2}\,s_{ij} \\ \\ d\sigma_{ij} &= 2G de_{ij} + K\,d\varepsilon_{kk}\,\delta_{ij} - \dfrac{G s_{mn} de_{mn}}{2k^2}\,s_{ij} \end{cases} \tag{7.54}$$

The product $s_{mn}de_{mn}$ of two deviators of stress and strain is called the rate work due to distortion. Decomposing the deviator of strain into elastic and plastic parts, we get:

$$s_{mn}de_{mn} = s_{mn}(de^e_{mn}de^p_{mn}) \tag{7.55}$$

where for elastic part $de^e_{mn} = ds_{mn}/2G$ (see (6.34)), then:

$$s_{mn}de_{mn} = s_{mn}(de^e_{mn} + de^p_{mn}) = (1/2G)\,s_{mn}\,ds_{mn} + s_{mn}de^p_{mn} = s_{mn}de^p_{mn} \tag{7.56}$$

because of $s_{mn}\,ds_{mn} = df = 0$ on the yield surface. Substituting $i = j$ into (7.54)) leads to:

$$d\varepsilon_{jj} = \frac{dI_\sigma}{3K} = d\varepsilon^e_{jj} \tag{7.57}$$

then:

$$d\varepsilon^p_{jj} = d\varepsilon_{jj} - d\varepsilon^e_{jj} = 0 \tag{7.58}$$

so the increment of the plastic strain is *incompressible*.

Hence, the following specifications are valid for an elastic-perfect plastic material obeying Huber-Mises criterion and flow rule:

- The increment of mean stress and mean strain obey the Hooke's law at all time:

$$d\sigma_{jj} = 3K\,d\varepsilon_{jj} \tag{7.59}$$

  and no plastic volume change can occurs $d\varepsilon^p_{jj} = 0$.

- The material is elastic satisfying the Hooke's law, meaning no change in increment of plastic strain $d\varepsilon^p_{ij} = 0$ as long as $\sqrt{II_s} < k$.

- Yielding occurs when and only when $\sqrt{II_s} = k$,

$$d\varepsilon^p_{ij} = \frac{s_{mn}\,de_{mn}}{2k^2}\,s_{ij} \tag{7.60}$$

- States when $\sqrt{II_s} > k$ can not exist for this material.

7.4 Effect of Strain Hardening on Yield Locus

For elastic perfect plastic materials presented in the last section, k is assumed constant, and the yield locus is fixed in stress space. For work-hardening we have two main models [6]. According to the *isotropic hardening* model, the effect of strain hardening is simply to expand the yield locus without changing its shape. In the *kinematic hardening* model, plastic deformation simply shifts the yield locus in the direction of the loading path without changing its shape or size (see Figure 7.10).

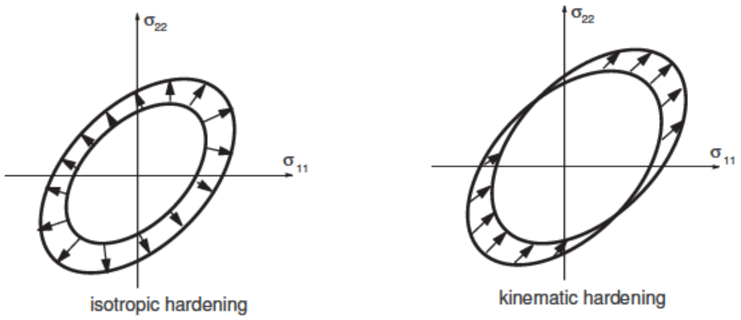

Figure 7.10:

Since now the yield locus is not fixed, then $d\sigma_{ij}$ is directed outward from f (loading) see Figure 7.11. To build a model of elastic plastic material with hardening we make

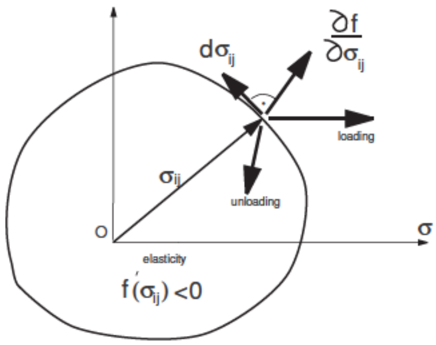

Figure 7.11:

the following basic assumptions: 1) The existence of the initial yield surface and subsequent loading surface; 2) The appropriate rule for describing the subsequent loading surfaces and 3) A flow rule to specify the stress-strain relation.

We go into the details, only for the case of *isotropic hardening*:

1. The yield locus (7.19) is the initial yield. Assuming a subsequent yield surface, which depends on the stress, on history of loading represented by the plastic strain ε^p_{ij} and a hardening parameter k:

$$f = f(\sigma_{ij}, \varepsilon^p_{ij}, k) \tag{7.61}$$

When $f = 0$ we have yield states, while $f < 0$ represents elastic states. On loading paths other than uniaxial tension, we need to specify the hardening parameter k. For this purpose, we introduce the concepts of effective stress σ_e and effective strain ε_e . They are defined as follows:

a) σ_e and ε_e reduce to σ and ε in tension test.

b) It is postulated that the strain hardening depends only on ε_e and there is a unique relation:

$$\sigma_e = \sigma_e(\varepsilon_e) \tag{7.62}$$

Because for a tension test σ_e and ε_e reduce to σ and ε (assumption a)), the σ-ε curve in a tension test is also the $\sigma_e - \varepsilon_e$ curve, so we can use the tension curve to predict the stress-strain behavior under other forms of loading.

When σ_e reaches the current flow stress, plastic deformation will occur, for the Huber-Mises criterion (7.36), the effective stress is:

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{\frac{1}{2}} \tag{7.63}$$

In terms of non-principal stresses, then:

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right]^{\frac{1}{2}} \tag{7.64}$$

There are two definitions of effective plastic strain depending on the form of k. For the *strain hardening hypothesis*, the function (7.61) takes the form:

$$f(\sigma_{ij}, \varepsilon_{ij}^p, k) = F(\sigma_{ij}, \varepsilon_{ij}^p) - k(\varepsilon_e^p) = 0 \tag{7.65}$$

where k is a monotonically increasing function, which depends only on the *effective plastic strain* ε_e^p defined as:

$$d\varepsilon_e^p = \sqrt{2/3\, d\varepsilon_{ij}^p\, d\varepsilon_{ij}^p}\ ;\ \varepsilon_e^p = \int d\varepsilon_e^p \tag{7.66}$$

In uniaxial loading $\varepsilon_e^p = \varepsilon^p$, where ε^p is the total uniaxial plastic strain.

For *work-hardening hypothesis*, we assume that k is a function of *total plastic work* W^p defined as:

$$dW^p = \sigma_e\, d\varepsilon_e^p\ ;\ W^p = \int dW^p \tag{7.67}$$

For the Huber-Mises criterion, (7.67) and (7.66) are equal (see for example (7.92) and (7.93)).

2. There exists a plastic potential function $g(\sigma_{ij}, \varepsilon_{ij}^p, k)$ so that the plastic strain could be derived from:

$$d\varepsilon_{ij}^p = d\lambda\, \frac{\partial g}{\partial \sigma_{ij}} \tag{7.68}$$

3. In plastic loading, both initial yield and subsequent stress states must satisfy the yield function $f(\sigma_{ij}, \varepsilon_{ij}^p, k) = 0$:

$$f = 0 \quad \text{and} \quad f + df = 0 \tag{7.69}$$

Hence, the *consistency condition*, which means that loading from a plastic state must lead to another plastic state, applied:

$$df = \frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}^p}\, d\varepsilon_{ij}^p + \frac{\partial f}{\partial k}\, dk = 0 \tag{7.70}$$

where the hardening parameter k is a function of plastic strain $k = k(\varepsilon_{ij}^p)$.

4. The total increment of the strain is the sum of the elastic and plastic part:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \tag{7.71}$$

where the increment of the elastic strain satisfies the Hooke's law:

$$d\sigma_{ij} = C_{ijkl}^e \, d\varepsilon_{kl}^e \tag{7.72}$$

From (7.71,(7.72) and (7.28) we obtain:

$$d\sigma_{ij} = C_{ijkl}^e \left(d\varepsilon_{kl} - d\varepsilon_{kl}^e\right) = C_{ijkl}^e \left(d\varepsilon_{kl} - d\lambda \frac{\partial g}{\partial \sigma_{kl}}\right) \tag{7.73}$$

Using (7.73), relation (7.70) takes the form:

$$\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl}^e \left(d\epsilon_{kl} - \frac{\partial g}{\partial \sigma_{kl}}\right) + \frac{\partial f}{\partial \varepsilon_{ij}^p} \, d\lambda \, \frac{\partial g}{\partial \sigma_{ij}} + \frac{\partial f}{\partial k} \, \frac{\partial k}{\partial \varepsilon_{ij}^p} \, d\lambda \, \frac{\partial g}{\partial \sigma_{ij}} = 0 \tag{7.74}$$

then we have:

$$d\lambda = \frac{(\partial f / \partial \sigma_{ij}) \, C_{ijkl}^e \, d\varepsilon_{kl}}{h + (\partial f / \partial \sigma_{mn}) \, C_{mnpq}^e (\partial g / \partial \sigma_{pq})} \tag{7.75}$$

where h is so-called the *hardening function* defined by:

$$h = -\frac{\partial f}{\partial \varepsilon_{ij}^p} \, \frac{\partial g}{\partial \sigma_{ij}} - \frac{\partial f}{\partial k} \, \frac{\partial k}{\partial \varepsilon_{ij}^p} \, \frac{\partial g}{\partial \sigma_{ij}} \tag{7.76}$$

Then the increment of plastic strain is given by:

$$d\varepsilon_{ij}^p = d\lambda \, \frac{\partial g}{\partial \sigma_{ij}} = \frac{(\partial f / \partial \sigma_{rs}) \, C_{rskl}^e \, d\varepsilon_{kl}}{h + (\partial f / \partial \sigma_{mn}) \, C_{mnpq}^e (\partial g / \partial \sigma_{pq})} \, \frac{\partial g}{\partial \sigma_{ij}} \tag{7.77}$$

The stress-strain relations for an elastic-work hardening plastic solid are:

$$d\sigma_{ij} = (C_{ijkl}^e + C_{ijkl}^p) \, d\varepsilon_{kl} \tag{7.78}$$

where:

$$C_{ijkl}^p = -\frac{C_{ijtu}^e \, (\partial f / \partial \sigma_{rs}) \, (\partial g / \partial \sigma_{tu}) \, C_{rskl}^e}{h + (\partial f / \partial \sigma_{mn}) \, C_{mnpq}^e \, (\partial g / \partial \sigma_{pq})} \tag{7.79}$$

In general $C_{ijkl}^p \neq C_{klij}^p$ because $f \neq g$. In the case when the plastic potential is the same as the yield function $f = g$, the flow rule (7.77) is called *associated flow rule*, we have:

$$C_{ijkl}^p = -\frac{C_{ijtu}^e \, (\partial f / \partial \sigma_{rs}) \, (\partial f / \partial \sigma_{tu}) \, C_{rskl}^e}{h + (\partial f / \partial \sigma_{mn}) \, C_{mnpq}^e \, (\partial f / \partial \sigma_{pq})} \tag{7.80}$$

The last tensor is symmetric.

Problem 20 *Show that the relation (7.75) reduces to the relation (7.53) for an isotropic elastic-perfect plastic material with associated flow rule.*

Solution: In this case from (7.76) we have $h = 0$ (the functions f and k do not depend upon ε^p_{ij}). For isotropic material, from (6.21) we have:

$$C^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{7.81}$$

then:

$$\frac{\partial f}{\partial\sigma_{ij}}\, C^e_{ijkl} = \lambda\delta_{kl}\left(\frac{\partial f}{\partial\sigma_{ij}}\,\delta_{ij}\right) + 2\mu\,\frac{\partial f}{\partial\sigma_{kl}}$$

By substituting this relation into the relation (7.75) and using the relation between the elastic moduli $\lambda = K - 2\mu/3$ (see Table 6.2), we get (7.53).

The *isotropic-hardening rule* assumes that the initial yield and subsequent loading surface throughout the deformation process are defined by:

$$f(\sigma_{ij}, k) = F(\sigma_{ij}) - k(\varepsilon^p_e) = 0 \tag{7.82}$$

so the initial yield expand without changing the shape. Here k is a scalar function of deformation history which defines the size of the current yield surface. It is a monotonically increasing function of a history parameter. This parameter can be taken to be the generalized plastic strain (7.66):

$$d\varepsilon^p_e = \sqrt{2/3\, d\varepsilon^p_{ij}\, d\varepsilon^p_{ij}} = \sqrt{\frac{2}{3}\left(d\lambda\frac{\partial g}{\partial\sigma_{ij}}\right)\left(d\lambda\frac{\partial g}{\partial\sigma_{ij}}\right)} = d\lambda\sqrt{\frac{2}{3}\,\frac{\partial g}{\partial\sigma_{ij}}\,\frac{\partial g}{\partial\sigma_{ij}}} \tag{7.83}$$

or with the total plastic work (7.67):

$$dW^p = \sigma_{ij}\, d\varepsilon^p_{ij} = s_{ij}\, d\varepsilon^p_{ij} = (d\lambda)\, s_{ij}\,\frac{\partial g}{\partial\sigma_{ij}} \tag{7.84}$$

where s_{ij} is the deviator of stress, because of the plastic incompressibility.

Problem 21 *Calculate the hardening function h in (7.75) for elastic-strain plastic hardening for an isotropic material satisfying the associated Huber-Mises flow rule* $f = \sqrt{II_s} - k = \sqrt{\frac{1}{2}\, s_{ij}\, s_{ij}} - k$ *in the case of linear hardening* $\sigma_Y = Y + E_1\,\varepsilon^p_e$ *(*Y *is the initial yield value, and* σ_Y *is the actual yield value, see Figure 7.12).*

Solution:

For the Huber-Mises yield criterion $f = \sqrt{II_s} - k$, we have $k(\varepsilon^p_e) = \sigma_Y(\varepsilon^p_e)/\sqrt{3}$, where $\sigma_Y(\varepsilon^p_e)$ is the tensile yield stress. We can find this from the stress-strain curve in tension (or compression). For linear hardening materials [5]:

$$\sigma_Y = Y + E_1\,\varepsilon^p_e \tag{7.85}$$

here E_1 is a material constant, then we have:

$$W^p = \left(Y + \frac{1}{2}\, E_1\,\varepsilon^p_e\right)\,\varepsilon^p_e \tag{7.86}$$

Then

$$\sigma^2_Y = Y^2 + 2\, E_1\left(Y + \frac{1}{2}\, E_1\,\varepsilon^p_e\right)\,\varepsilon^p_e = Y^2 + 2\, E_1\, W^p \tag{7.87}$$

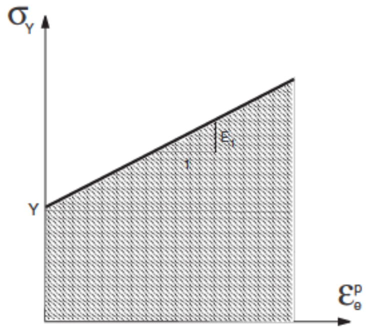

Figure 7.12:

We have:

$$\frac{\partial f}{\partial k}\frac{\partial k}{\partial \varepsilon^p_e} = -\frac{1}{\sqrt{3}}\frac{\partial \sigma_Y}{\partial \varepsilon^p_e} = -\frac{E_1}{\sqrt{3}} \tag{7.88}$$

in the case where k is a function of the generalized plastic strain (7.66), and:

$$\frac{\partial f}{\partial k}\frac{\partial k}{\partial W^p} = -\frac{1}{\sqrt{3}}\frac{\partial \sigma_Y}{\partial W^p} = -\frac{E_1}{\sqrt{3}\,\sigma_Y} \tag{7.89}$$

in the case where k is a function of the total plastic work (7.67).

The increment of k can be written in different forms:

$$dk = \frac{\partial k}{\partial \varepsilon^p_{ij}}\,d\varepsilon^p_{ij} = \frac{\partial k}{\partial \varepsilon^p_e}\,d\varepsilon^p_e = \frac{\partial k}{\partial W^p}\,dW^p \tag{7.90}$$

By using (7.83), (7.84) the expression (7.76) for the hardening function h is:

$$\begin{aligned} h &= -\frac{\partial f}{\partial \varepsilon^p_{ij}}\frac{\partial g}{\partial \sigma_{ij}} - \frac{\partial f}{\partial k}\frac{\partial k}{\partial \varepsilon^p_{ij}}\frac{\partial g}{\partial \sigma_{ij}} \\ &= -\frac{\partial f}{\partial \varepsilon^p_{ij}}\frac{\partial g}{\partial \sigma_{ij}} - \frac{\partial f}{\partial k}\frac{\partial k}{\partial \varepsilon^p_e}\sqrt{\frac{2}{3}}\sqrt{\frac{\partial g}{\partial \sigma_{mn}}\frac{\partial g}{\partial \sigma_{mn}}} \\ &= -\frac{\partial f}{\partial \varepsilon^p_{ij}}\frac{\partial g}{\partial \sigma_{ij}} - \frac{\partial f}{\partial k}\frac{\partial k}{\partial W^p}\,s_{ij}\,\frac{\partial g}{\partial \sigma_{ij}} \end{aligned} \tag{7.91}$$

In this problem: $g = f$; $\partial f/\partial k = -1$; $\partial f/\partial \varepsilon^p_{ij} = 0$; $(\partial f/\partial \sigma_{ij}) = s_{ij}/2\sqrt{II_s}$ (see (7.52)). By substituting (7.88) and (7.89) into (7.91) we have finally:

$$h = -\frac{\partial f}{\partial k}\frac{\partial k}{\partial \varepsilon^p_e}\sqrt{\frac{2}{3}}\sqrt{\frac{\partial f}{\partial \sigma_{mn}}\frac{\partial f}{\partial \sigma_{mn}}} = \frac{E_1}{\sqrt{3}}\sqrt{\frac{2}{3}}\sqrt{\frac{\partial f}{\partial \sigma_{mn}}\frac{\partial f}{\partial \sigma_{mn}}} = \frac{E_1}{\sqrt{3}}\sqrt{\frac{2}{3}}\sqrt{\frac{s_{ij}}{2k}\frac{s_{ij}}{2k}} = \frac{E_1}{3} \tag{7.92}$$

in the case where k is a function of the generalized plastic strain (7.66), and

$$h = -\frac{\partial f}{\partial k}\frac{\partial k}{\partial W^p}\,s_{ij}\,\frac{\partial f}{\partial \sigma_{ij}} = \frac{E_1}{\sqrt{3}\,\sigma_Y}\,s_{ij}\,\frac{\partial f}{\partial \sigma_{ij}} = \frac{E_1}{\sqrt{3}\,\sigma_Y}\frac{s_{ij}\,s_{ij}}{2k} = \frac{E_1}{\sqrt{3}\,\sigma_Y}\frac{2k^2}{2k} = \frac{E_1}{3} \tag{7.93}$$

in the case where $k = \sigma_Y/\sqrt{3}$ is a function of the total plastic work (7.67). Hence, in the case of the Huber-Mises criterion, the two definitions of effective plastic strain lead to identical results.

Problem 22 *Study plane bending of a rectangular cross-section beam made of a material with elastic perfectly plastic behaviour (see Figure 7.13), in the elasto-plastic regime.*

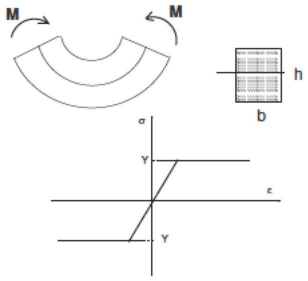

Figure 7.13:

Solution:

Considering the rectangular cross-section represented in Fig. 7.13, in this case the neutral axis divides the cross-section in two equal parts, since the material behaviour is the same for compressive and tensile stresses. In elastic bending, the distribution of the normal stress in the cross section is linear:

$$\sigma_{xx} = -Mz/J \tag{7.94}$$

where M is the bending moment, J is the inertia moment of the cross-section with respect to the neutral axis.

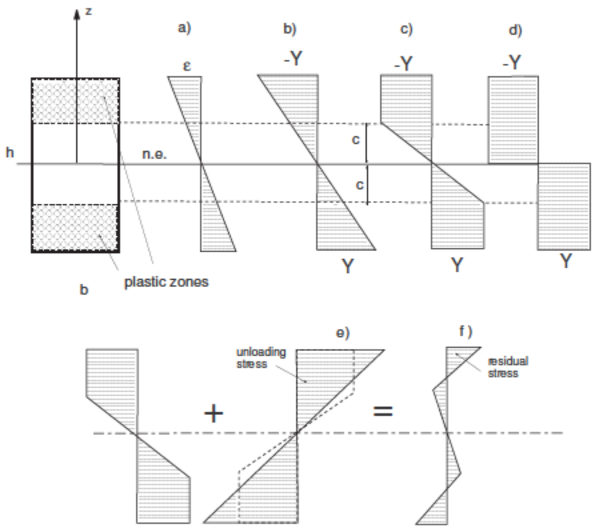

Figure 7.14:

If the bending moment exceeds the value corresponding to the yielding strain in the fibres farthest from the neutral axis (Fig. 7.14b):

$$M_Y = \frac{Ybh^2}{6} \tag{7.95}$$

which is the highest possible bending moment in the elastic phase, the fibres undergoing more strain yield and the bar enters in the elasto-plastic regime (Fig. 7.14c). In the cross-section there are elastic and plastic zones. The value of c (with $2c$ being the height of the part of the section still under elastic deformations) corresponding to $M > M_T$ can be found from the relation for bending moment:

$$\int_{-h/2}^{h/2} \sigma\, z\, b\, dz = M \tag{7.96}$$

In the plastic zones, $\sigma = Y$ and in the fibres which are still under elastic deformation, we use (7.94) obtaining:

$$M = \frac{Yb}{12}\left(3h^2 - 4c^2\right) \tag{7.97}$$

By making $c = 0$ in (7.97) we get the yielding bending moment M_p, where yielding of the entire cross-section takes place:

$$M_p = \frac{Ybh^2}{4} \tag{7.98}$$

With bending moment $M > M_p$, the curvature of the bar may then be increased practically infinitely without any increase in the bending moment.

If the bar is unloaded after the maximum bending moment in the elastic phase is exceeded, the internal stresses do not disappear totally, since the material behaves elastically in the unloading process (Fig.7.14e) and some residual deformation is left (see Fig. 7.14f) in the fibres where the yielding strain was exceeded (we have applied the superposition of a linear elastic diagram on the elasto-plastic diagram resulting from the loading to obtain Fig. 7.14f).

Problem 23 *Find a solution for a hollow cylinder of inner radius a and outer radius b is subjected to an external traction p. The material is elastic-perfectly plastic with the Huber-Mises criterion with yield value Y.*

Solution:

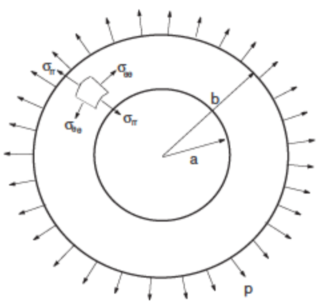

Figure 7.15:

1. **Elastic state** $(0 < p < p_{pl})$

   When the magnitude of the traction is smaller than a value p_{pl}, now unknown:

$$0 < p_e < p_{pl} \tag{7.99}$$

the material obeys Hooke's law, the use of the general solution (6.190) with the outer traction $p_o = -p$ leads to:

$$\sigma_{rr} = \frac{b^2\, p}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) \tag{7.100}$$

$$\sigma_{\vartheta\vartheta} = \frac{b^2\, p}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) \tag{7.101}$$

We observe that the radial component of the stress is always smaller than the tangential component: $0 \leq \sigma_{rr} < \sigma_{\vartheta\vartheta}$ for every $a \leq r \leq b$. The latter is greatest at the inner surface of the cylinder $r = a$.

2. **Elastic-plastic state** $(p_{pl} \leq p < p_{max})$

When p grows, as long as the state stress does not satisfy the yield criterion:

$$\sigma_{rr}^2 - \sigma_{rr}\,\sigma_{\vartheta\vartheta} + \sigma_{\vartheta\vartheta}^2 = Y^2 \tag{7.102}$$

anywhere in the cylinder, it is in elastic state. However, as p is increased beyond a value $p = p_{pl}$, yield is initiated. To find this value, substituting (7.100) and (7.101) into (7.102), we find that:

$$\frac{p_{pl}}{Y} = \frac{1}{2}\left(1 - \frac{a^2}{b^2}\right) \tag{7.103}$$

and yield is initiated at the inner surface $r = a$ and spreads toward the outer surface $r = b$. Hence, when $p_{pl} \leq p$, but still lower than a critical value p_{max}, both elastic and plastic zone exist (see Figure 7.16). Denote by r_{pl} the boundary between elastic and plastic zones.

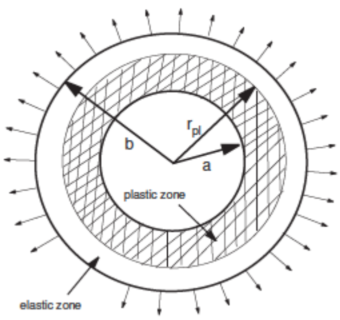

Figure 7.16:

From equation (7.102) we have:

$$\sigma_{\vartheta\vartheta} = \frac{1}{2}\left(\sigma_{rr} + \sqrt{4\,Y^2 - 3\,\sigma_{rr}^2}\right) \tag{7.104}$$

with the condition $4Y^2 - 3\,\sigma_{rr}^2 \geq 0$, or:

$$\frac{\sigma_{rr}}{Y} \leq \frac{2}{\sqrt{3}} = 1,155 \qquad (*)$$

In plastic zone, material must obey the yield function (7.102) and the equation of equilibrium, now in the form (see (6.182)):

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\vartheta\vartheta}}{r} = 0 \tag{7.105}$$

Substituting (7.104) into (7.105), and integrating we obtain:

$$r = \mathcal{C} \sqrt{\frac{2}{\sqrt{3}\left(-\sigma_{rr} + \sqrt{4\,Y^2 - 3\,\sigma_{rr}^2}\right)}} \exp\left\{-\sqrt{3}\ \mathrm{arctg}\sqrt{\frac{2Y - \sqrt{3}\,\sigma_{rr}}{2Y + \sqrt{3}\,\sigma_{rr}}}\right\} \tag{7.106}$$

with condition

$$\frac{\sigma_{rr}}{Y} < 1 \qquad (**)$$

The integration constant $\mathcal{C}$ can be found from the boundary condition $\sigma_{rr} = 0$ at $r = a$:

$$a = \mathcal{C} \sqrt{\frac{1}{\sqrt{3}\,Y}} \exp\left(-\sqrt{3}\,\mathrm{arctg}\,1\right) \text{ then } \mathcal{C} = a\,\sqrt[4]{3}\,\sqrt{Y}\,\exp\left(\frac{\sqrt{3}\,\pi}{4}\right) \tag{7.107}$$

then in the plastic zone, the component of stress σ_{rr} is follows:

$$\frac{r}{a} = \sqrt{\frac{2\,Y}{-\sigma_{rr} + \sqrt{4Y^2 - 3\,\sigma_{rr}^2}}} \exp\left\{\frac{\sqrt{3}\,\pi}{4} - \sqrt{3}\ \mathrm{arctg}\sqrt{\frac{2Y - \sqrt{3}\,\sigma_{rr}}{2Y + \sqrt{3}\,\sigma_{rr}}}\right\} \tag{7.108}$$

Having σ_{rr}, we can find the tangential component $\sigma_{\vartheta\vartheta}$ in the plastic zone using the relation (7.104). By continuity, the components of the stress must be equal in the boundary $r = r_{pl}$, then we have the following system of equations to find the value of radial stress on the boundary between elastic and plastic zones q and r_{pl}:

$$\begin{cases} \dfrac{r_{pl}}{a} = \sqrt{\dfrac{2\,Y}{-q + \sqrt{4Y^2 - 3\,q^2}}} \exp\left\{\dfrac{\sqrt{3}\,\pi}{4} - \sqrt{3}\ \mathrm{arctg}\sqrt{\dfrac{2Y - \sqrt{3}\,q}{2Y + \sqrt{3}\,q}}\right\} \\[2ex] \dfrac{1}{2}\left(q + \sqrt{4\,Y^2 - 3\,q^2}\right) = \dfrac{p\,b^2 - q\,r_{pl}^2}{b^2 - r_{pl}^2} - \dfrac{b^2(q-p)}{b^2 - r_{pl}^2} \end{cases} \tag{7.109}$$

3. **Maximal load** $p = p_{max}$

When p reaches the value p_{max}, the plastic zone spreads to $r = b$, so $\sigma_{rr} = p_{max}$ at $r_{pl} = b$, we have a following equation to find p_{max}:

$$\frac{b}{a} = \sqrt{\frac{2\,Y}{-p_{max} + \sqrt{4Y^2 - 3\,p_{max}^2}}} \exp\left\{\frac{\sqrt{3}\,\pi}{4} - \sqrt{3}\ \mathrm{arctg}\sqrt{\frac{2Y - \sqrt{3}\,p_{max}}{2Y + \sqrt{3}\,p_{max}}}\right\} \tag{7.110}$$

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